Invariant vector fields and the prolongation method for supersymmetric quantum systems

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Abstract

The kinematical and dynamical symmetries of equations describing the time evolution of quantum systems like the supersymmetric harmonic oscillator in one space dimension and the interaction of a non-relativistic spin one-half particle in a constant magnetic field are reviewed from the point of view of the vector field prolongation method. Generators of supersymmetries are then introduced so that we get Lie superalgebras of symmetries and supersymmetries. This approach does not require the introduction of Grassmann valued differential equations but a specific matrix realization and the concept of dynamical symmetry. The Jaynes-Cummings model and supersymmetric generalizations are then studied. We show how it is closely related to the preceding models. Lie algebras of symmetries and supersymmetries are also obtained.

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1 Introduction

The symmetries of a system of ordinary differential equations (ODEs) or partial differential equations (PDEs) are usually obtained by using the so-called prolongation method of vector fields [1, 2]. It consists of finding the infinitesimal generators which close the maximal invariant Lie algebra of the system of equations. The corresponding symmetry group is the Lie group of local transformations of independent and dependent variables which leaves invariant the system under consideration. Such a system may be associated with the wave equation of some quantum model. The independent variables are the usual space-time coordinates while the dependent ones are the components of the wave function. The symmetries may be related to the so-called kinematical Lie algebra [3] of the quantum system.

Now, if we have in mind supersymmetric (SUSY) quantum models [4, 5], the question is how to find them from this prolongation method. We answer this question by considering first some standard examples where the kinematical Lie superalgebras are known. This is the case of the SUSY harmonic oscillator in one space dimension (see [6, 7] and reference therein) and the Pauli equation, in two space dimensions, describing the motion of a non-relativistic spin one-half particle in a constant magnetic field [7]. An important part of this work is concerned by the study of symmetries of the Jaynes-Cummings (JC) model [8] based on the same approach. Let us recall that the JC model, which consists of an idealized description of the interaction of a quantized electromagnetic field and an atomic system with two levels, is closely related with the two models considered before. An interesting point is that it can be made SUSY in a non-trivial manner and our approach will clarify this point and will make the connection with different works on this subject [9, 10].

At the classical level, Grassmann-valued differential equations have been introduced [11, 12, 13, 14] and the prolongation method has been extended to include Grassmann independent and dependent variables [15, 16]. For example, SUSY extensions of Korteweg-de-Vries and other equations have been studied and maximal invariant Lie superalgebras have been obtained.

At the quantum level, the problem is somewhat different. The SUSY system is nothing

but a set of PDEs with the usual independent and dependent variables. So it is really of the type where the usual prolongation method can be used and the vector fields obtained close a Lie algebra. The non-trivial question we ask is how to get the generators which are associated with supersymmetries from this method and which, together with the symmetry generators, close a Lie superalgebra.

To clarify the context we are working with, let us here recall the prolongation method [1, 2] for determining the symmetries of a system of m PDEs of order n of the type

$$\Delta^{(k)}[x; u_\alpha, u^{(1)}_{\alpha x_{j_1}}, u^{(2)}_{\alpha x_{j_1} x_{j_2}}, \cdots, u^{(n)}_{\alpha x_{j_1} x_{j_2} \cdots x_{j_n}}] = 0, \qquad k = 1, 2, \ldots, m, \tag{1}$$

with p independent variables x_j $(j = 1, 2, \ldots p)$, and q dependent variables $u_\alpha(x)$ $(\alpha = 1, 2, \ldots q)$. The derivatives of the dependent variables are defined as

$$u^{(l)}_{\alpha x_{j_1} x_{j_2} \cdots x_{j_l}} \equiv u^{(l)}_\alpha \equiv \frac{\partial^l u_\alpha(x)}{\partial x_{j_1} \partial x_{j_2} \cdots \partial x_{j_l}}, \qquad 1 \leq l \leq n, \tag{2}$$

where the integers j_r $(r = 1, 2, \ldots, l)$ are such that $0 \leq j_r \leq p$.

The Lie group of local transformations of independent and dependent variables which leave invariant such a system is obtained by performing the following infinitesimal transformation on the independent and dependent variables:

$$\tilde{x}_j = x_j + \epsilon \sum_j \xi_j(x, u_\alpha) + \mathcal{O}(\epsilon^2), \tag{3}$$

$$\tilde{u}_\alpha(\tilde{x}, \tilde{u}_\beta) = u_\alpha(x, u_\beta) + \epsilon \phi_\alpha(x, u_\beta) + \mathcal{O}(\epsilon^2), \tag{4}$$

Assuming that they satisfy, at first order in ϵ, the equation

$$\Delta^{(k)}[\tilde{x}; \tilde{u}_\alpha, \tilde{u}^{(1)}_{\alpha \tilde{x}_{j_1}}, \tilde{u}^{(2)}_{\alpha \tilde{x}_{j_1} \tilde{x}_{j_2}}, \cdots, \tilde{u}_{\alpha^{(n)}_{\tilde{x}_{j_1} \tilde{x}_{j_2} \cdots \tilde{x}_{j_n}}}] = 0, \tag{5}$$

for $k = 1, 2, \ldots, m$ and when the $u_\alpha(x)$ solve the system (1), we can find the functions ξ_j and ϕ_α. A practical way to do it is to introduce the vector field

$$v = \sum_{j=1}^{p} \xi_j(x, u_\beta) \partial_{x_j} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u_\beta) \partial_{u_\alpha}, \tag{6}$$

associated with the transformations (3) and (4) and define the nth order prolongation of v as

$$pr^{(n)} v = v + \sum_{\alpha=1}^{q} \sum_{J_l,\, l=1,2,\ldots,n} \phi^{J_l}_\alpha(x, u_\beta, u^{(1)}_\beta, u^{(2)}_\beta, \ldots, u^{(n)}_\beta) \partial_{u^{J_l}_\alpha}, \tag{7}$$

where $J_l = (x_{j_1}, x_{j_2}, \ldots, x_{j_l})$ is the multi-index notation for the differentiation with respect to the x_j and $\partial_{u_\alpha^{J_l}} \equiv \partial_{u_\alpha^{(l)}}$. Note that the coefficients $\phi_\alpha^{J_l}$ satisfy the following recurrence relation

$$\phi_\alpha^{J_l, x_k} = D_{x_k} \phi_\alpha^{J_l} - \sum_{j=1}^{p} (D_{x_k} \xi_j) \frac{\partial u_\alpha^{J_l}}{\partial x_j}, \tag{8}$$

where D_{x_k} is the total derivative with respect to x_k. The infinitesimal criterion for invariance (5) may then be written as

$$pr^{(n)} v \{ \Delta^{(k)} [x; u_\alpha, u_{\alpha x_{j_1}}^{(1)}, u_{\alpha x_{j_1} x_{j_2}}^{(2)}, \cdots, u_{\alpha x_{j_1} x_{j_2} \cdots x_{j_n}}^{(n)}] \} = 0, \qquad k = 1, 2, \ldots, m, \tag{9}$$

when the $u_\alpha(x)$ satisfy (1). The condition (9) gives a set of PDE's called the determining equations which can be solved to get the explicit form of the functions ξ_j and ϕ_α in (6). The resolution may lead to different possibilities: no nontrivial solutions, a finite number of integration constants or that the general solution depends on arbitrary functions. Let us also mention that we have the following properties of the vector field prolongations:

$$pr^{(n)} (c_1 v_1 + c_2 v_2 + \cdots + c_n v_m) = pr^{(n)} c_1 v_1 + pr^{(n)} c_2 v_2 + \cdots + pr^{(n)} c_n v_m \tag{10}$$

and

$$pr^{(n)} [v_1, v_2] = [pr^{(n)} v_1, pr^{(n)} v_2]. \tag{11}$$

The contents of the paper is thus described as follows. Section 2 is devoted to the construction of invariant vector fields for the SUSY harmonic oscillator in one dimension. It admits a large set of symmetries and the integration of vector fields gives a matrix realization of the symmetry generators which is essential in order to find the generators of supersymmetries. The corresponding kinematical and dynamical invariance superalgebras will be recovered in this context. In Section 3, the model of a non-relativistic spin-$\frac{1}{2}$ particle in a constant magnetic field is studied. It can be reduced to a two dimensional model and shows a similar behaviour than the SUSY harmonic oscillator. The symmetry algebra and superalgebra are obtained from the prolongation of vector fields method and connected to the preceding case. In Section 4, we start with a quantum evolution equation which is a realization of the JC model and determine the invariant vector fields and the associated invariant algebra. The connection with the preceding models is very helpful to get a Lie superalgebra of symmetries for a generalized JC model. In section

5, we propose a SUSY version of this model and give the corresponding symmetries and supersymmetries. We also make the connection with preceding attemps to get SUSY JC models.

2 The SUSY harmonic oscillator

The first set of equations we are considering is the one associated with the SUSY harmonic oscillator in one space dimension. The corresponding Schrödinger evolution equation is

$$(i\partial_t - H_{\mathrm{SUSY}})\,\Psi(t,x) = 0. \tag{12}$$

Let us mention that along this work we use the convention that $\hbar = 1$. The SUSY Hamiltonian [17] is given by:

$$H_{\mathrm{SUSY}} = \Big(-\frac{1}{2M}\frac{\partial^2}{\partial x^2} + \frac{1}{2}M\omega^2 x^2\Big)\sigma_0 - \frac{\omega}{2}\sigma_3, \tag{13}$$

where σ_0 is the identity matrix and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The wave function takes the form

$$\Psi(t,x) = \begin{pmatrix} \psi_1(t,x) \\ \psi_2(t,x) \end{pmatrix}, \qquad \psi_1, \psi_2 \in L^2(\mathbb{R}). \tag{14}$$

It is convenient to write the equation (12) as a set of two equations

$$i\frac{\partial \psi_\alpha}{\partial t} + \frac{1}{2M}\frac{\partial^2 \psi_\alpha}{\partial x^2} - \frac{1}{2}M\omega^2 x^2 \psi_\alpha + \frac{\omega_\alpha}{2}\psi_\alpha = 0, \qquad \alpha = 1, 2, \tag{15}$$

where we have set $\omega_1 = \omega$ et $\omega_2 = -\omega$.

The kinematical and dynamical symmetries and supersymmetries have been largely studied [3, 5, 6, 7, 18] but these approaches were different than the one we want to apply. Indeed for the usual harmonic oscillator, Niederer [3] has first shown that the maximal kinematical algebra is the semi-direct sum $so(2,1) \dotplus h(2)$, where $h(2)$ is the usual Heisenberg-Weyl algebra. The maximal dynamical algebra [18], defined as the one associated with the degeneracy group of the model, is given by $sp(2) \dotplus h(2)$ and includes the preceding kinematical algebra. The dynamical and kinematical superalgebras of the SUSY version coincide in this one-dimensional case and is given by $osp(2/2) \dotplus sh(2/2)$ [6, 7]. We will show how to recover these structures starting from the prolongation method of vector fields applied to the system (15).

2.1 Prolongation method and invariant vector fields

A standard way of applying the prolongation method to a system containing complex valued functions is to express the components of the wave function (14) as

$$\psi_1(t,x) = u_1(t,x)e^{i\nu_1(t,x)}, \quad \psi_2(t,x) = u_2(t,x)e^{i\nu_2(t,x)}, \tag{16}$$

where $u_1,\ u_2,\ \nu_1$ and ν_2 are real functions of t and x. Inserting (16) into (15) and separating the real and complex parts of the resulting equations, we are led to a set of four coupled equations in $u_1,\ u_2,\ \nu_1$ and ν_2. The vector field (6) may be written explicitly as

$$v = \xi_1\partial_t + \xi_2\partial_x + \phi_1\partial_{u_1} + \phi_2\partial_{u_2} + \varphi_1\partial_{\nu_1} + \varphi_2\partial_{\nu_2}, \tag{17}$$

where ξ_j $(j=1,2)$, ϕ_α and φ_α $(\alpha=1,2)$ are real functions which depend on $t,\ x,\ u_1,\ u_2,\ \nu_1$ and ν_2.

A simpler way of solving the problem is to consider the set (15) together with its complex conjugated

$$-i\frac{\partial\bar{\psi}_\alpha}{\partial t} + \frac{1}{2M}\frac{\partial^2\bar{\psi}_\alpha}{\partial x^2} - \frac{1}{2}M\omega^2x^2\bar{\psi}_\alpha + \frac{\omega_\alpha}{2}\bar{\psi}_\alpha = 0, \qquad \alpha = 1,2. \tag{18}$$

Now the corresponding vector field takes the form

$$v = \xi_1\partial_t + \xi_2\partial_x + \Phi_1\partial_{\psi_1} + \bar{\Phi}_1\partial_{\bar{\psi}_1} + \Phi_2\partial_{\psi_2} + \bar{\Phi}_2\partial_{\bar{\psi}_2}, \tag{19}$$

where now ξ_j $(j=1,2)$ are real functions of the variables $t,\ x,\ \psi_1,\ \psi_2,\ \bar{\psi}_1$ and $\bar{\psi}_2$ and $\Phi_\alpha,\ \bar{\Phi}_\alpha$ $(\alpha=1,2)$ are possible complex valued functions of these variables. In terms of these variables, the second order prolongation of v takes the form:

$$pr^{(2)}v = v + \sum_{\alpha=1}^{2}\left(\Phi_\alpha^t\partial_{\psi_{\alpha,t}} + \Phi_\alpha^x\partial_{\psi_{\alpha,x}} + \Phi_\alpha^{tt}\partial_{\psi_{\alpha,tt}} + \Phi_\alpha^{tx}\partial_{\psi_{\alpha,tx}} + \Phi_\alpha^{xx}\partial_{\psi_{\alpha,xx}}\right) + [\text{c.c}], \tag{20}$$

where, for example, $\psi_{\alpha,t}$ is the usual partial derivative of ψ_α with respect to t. Applying this prolongation to the system consisting in equations (15) and (18), we get

$$i\Phi_\alpha^t + \frac{1}{2M}\Phi_\alpha^{xx} - \frac{1}{2M}\omega^2x^2\Phi_\alpha + \frac{\omega_\alpha}{2}\Phi_\alpha - M\omega^2x\xi_1\psi_\alpha = 0, \tag{21}$$

$$-i\bar{\Phi}_\alpha^t + \frac{1}{2M}\bar{\Phi}_\alpha^{xx} - \frac{1}{2M}\omega^2x^2\bar{\Phi}_\alpha + \frac{\omega_\alpha}{2}\bar{\Phi}_\alpha - M\omega^2x\xi_1\bar{\psi}_\alpha = 0, \tag{22}$$

where we have

$$\Phi_\alpha^t = D_t\Phi_\alpha - (D_t\xi_1)\,\psi_{\alpha,t} - (D_t\xi_2)\,\psi_{\alpha,x}, \tag{23}$$

$$\Phi_\alpha^{xx} = (D_x \Phi_\alpha^x) - (D_x \xi_1)\psi_{\alpha,xt} - (D_x \xi_2)\psi_{\alpha,xx}, \tag{24}$$

with

$$\Phi_\alpha^x = D_x \Phi_\alpha - (D_x \xi_1)\,\psi_{\alpha,t} - (D_x \xi_2)\,\psi_{\alpha,x}, \tag{25}$$

together with their complex conjugated and for $\alpha = 1, 2$. Inserting these expressions into the system (21-22), taking into account the equations (15) and (18) and identifying to zero the coefficients of the partial derivatives, we get a set of determining equations which will give the functions ξ_j, Φ_α and $\bar{\Phi}_\alpha$. Solving these equations, we get:

$$\begin{aligned}
\xi_1(t) &= \frac{1}{2\omega}(\delta_1 \sin 2\omega t - \delta_2 \cos 2\omega t) + \delta_3, & (26)\\
\xi_2(t,x) &= \frac{1}{2}(\delta_1 \cos 2\omega t + \delta_2 \sin 2\omega t)x + \delta_4 \cos \omega t + \delta_5 \sin \omega t, & (27)
\end{aligned}$$

which are effectively real functions depending only on the coordinates t and x. We also have

$$\begin{aligned}
\Phi_1(t,x,\psi_1,\psi_2) &= A_0(t,x) + A_1(t,x)\psi_1 + A_2(t)\psi_2, & (28)\\
\Phi_2(t,x,\psi_1,\psi_2) &= B_0(t,x) + B_1(t)\psi_1 + B_2(t,x)\psi_2, & (29)
\end{aligned}$$

where

$$\begin{aligned}
A_1(t,x) &= -\frac{1}{4}\left(e^{-2i\omega t} + 2iM\omega x^2 \sin 2\omega t\right)\delta_1 - \frac{i}{4}\left(e^{-2i\omega t} - 2M\omega x^2 \cos 2\omega t\right)\delta_2 \\
&- iM\omega x(\delta_4 \sin \omega t - \delta_5 \cos \omega t) + \delta_{13} + i\delta_6, & (30)\\
A_2(t) &= (\delta_7 - i\delta_{10})e^{i\omega t}, & (31)\\
B_1(t) &= (\delta_8 - i\delta_{11})e^{-i\omega t}, & (32)\\
B_2(t,x) &= -\frac{1}{4}\left(e^{2i\omega t} + 2iM\omega x^2 \sin 2\omega t\right)\delta_1 + \frac{i}{4}\left(e^{2i\omega t} + 2M\omega x^2 \cos 2\omega t\right)\delta_2 \\
&- iM\omega x(\delta_4 \sin \omega t - \delta_5 \cos \omega t) + \delta_9 + i\delta_{12}. & (33)
\end{aligned}$$

The parameters δ_i $(i = 1, 2, \ldots, 13)$ are all real and the functions $\bar{\Phi}_1,\ \bar{\Phi}_2$ are in fact the complex conjugate of $\Phi_1,\ \Phi_2$. The functions $A_0(t,x),\ B_0(t,x)$ and their conjugated $\bar{A}_0(t,x),\ \bar{B}_0(t,x)$ are such that they satisfy respectively (15) and (18) for $\psi_1 = A_0$ and $\psi_2 = B_0$.

The infinitesimal generators of the invariance finite dimensional Lie algebra are thus easily obtained using the preceding equations and (19). We get

$$\tilde{X}_1 = \frac{1}{2\omega}\sin 2\omega t \partial_t + \frac{x}{2}\cos 2\omega t \partial_x - \frac{1}{4}\cos 2\omega t(\psi_1 \partial_{\psi_1} + \bar{\psi}_1 \partial_{\bar{\psi}_1} + \psi_2 \partial_{\psi_2} + \bar{\psi}_2 \partial_{\bar{\psi}_2})$$

$$
\begin{aligned}
& - && i\frac{M\omega x^2}{2}\sin 2\omega t\left((\psi_1\partial_{\psi_1}-\bar\psi_1\partial_{\bar\psi_1})+(\psi_2\partial_{\psi_2}-\bar\psi_2\partial_{\bar\psi_2})\right)\\
& + && \frac{i}{4}\sin 2\omega t\left((\psi_1\partial_{\psi_1}-\bar\psi_1\partial_{\bar\psi_1})-(\psi_2\partial_{\psi_2}-\bar\psi_2\partial_{\bar\psi_2})\right),\\
\tilde X_2 & = && -\frac{1}{2\omega}\cos 2\omega t\partial_t+\frac{x}{2}\sin 2\omega t\partial_x-\frac{1}{4}\sin 2\omega t(\psi_1\partial_{\psi_1}+\bar\psi_1\partial_{\bar\psi_1}+\psi_2\partial_{\psi_2}+\bar\psi_2\partial_{\bar\psi_2})\\
& + && i\frac{M\omega x^2}{2}\cos 2\omega t\left((\psi_1\partial_{\psi_1}-\bar\psi_1\partial_{\bar\psi_1})+(\psi_2\partial_{\psi_2}-\bar\psi_2\partial_{\bar\psi_2})\right)\\
& - && \frac{i}{4}\cos 2\omega t\left((\psi_1\partial_{\psi_1}-\bar\psi_1\partial_{\bar\psi_1})-(\psi_2\partial_{\psi_2}-\bar\psi_2\partial_{\bar\psi_2})\right),\\
\tilde X_3 & = && \partial_t,\\
\tilde X_4 & = && \cos\omega t\partial_x-iM\omega x\sin\omega t\left((\psi_1\partial_{\psi_1}-\bar\psi_1\partial_{\bar\psi_1})+(\psi_2\partial_{\psi_2}-\bar\psi_2\partial_{\bar\psi_2})\right),\\
\tilde X_5 & = && \sin\omega t\partial_x+iM\omega x\cos\omega t\left((\psi_1\partial_{\psi_1}-\bar\psi_1\partial_{\bar\psi_1})+(\psi_2\partial_{\psi_2}-\bar\psi_2\partial_{\bar\psi_2})\right),\\
\tilde X_6 & = && i(\psi_1\partial_{\psi_1}-\bar\psi_1\partial_{\bar\psi_1}),\\
\tilde X_7 & = && e^{i\omega t}\psi_2\partial_{\psi_1}+e^{-i\omega t}\bar\psi_2\partial_{\bar\psi_1},\\
\tilde X_8 & = && e^{-i\omega t}\psi_1\partial_{\psi_2}+e^{i\omega t}\bar\psi_1\partial_{\bar\psi_2},\\
\tilde X_9 & = && \psi_2\partial_{\psi_2}+\bar\psi_2\partial_{\bar\psi_2},\\
\tilde X_{10} & = && i\left(e^{-i\omega t}\bar\psi_2\partial_{\bar\psi_1}-e^{i\omega t}\psi_2\partial_{\psi_1}\right),\\
\tilde X_{11} & = && i\left(e^{i\omega t}\bar\psi_1\partial_{\bar\psi_2}-e^{-i\omega t}\psi_1\partial_{\psi_2}\right),\\
\tilde X_{12} & = && i(\psi_2\partial_{\psi_2}-\bar\psi_2\partial_{\bar\psi_2}),\\
\tilde X_{13} & = && (\psi_1\partial_{\psi_1}+\bar\psi_1\partial_{\bar\psi_1}).
\end{aligned}
$$

If we come back to the real variables u_α et ν_α ($\alpha=1,2$) introduced in (16), we have the following correspondence:

$$
\partial_{\psi_\alpha}=\frac{e^{-i\nu_\alpha}}{2}\left(\partial_{u_\alpha}-\frac{i}{u_\alpha}\partial_{\nu_\alpha}\right),\quad \partial_{\bar\psi_\alpha}=\frac{e^{i\nu_\alpha}}{2}\left(\partial_{u_\alpha}+\frac{i}{u_\alpha}\partial_{\nu_\alpha}\right),\quad \alpha=1,2. \tag{34}
$$

For example, we can write

$$
(\psi_\alpha\partial_{\psi_\alpha}+\bar\psi_\alpha\partial_{\bar\psi_\alpha})=u_\alpha\partial_{u_\alpha},\qquad i(\psi_\alpha\partial_{\psi_\alpha}-\bar\psi_\alpha\partial_{\bar\psi_\alpha})=\partial_{\nu_\alpha},\quad \alpha=1,2. \tag{35}
$$

So from equations (16), (34) and after a slight change of basis, we get the following generators:

$$
\begin{aligned}
X_1 & = && \frac{1}{2\omega}\sin 2\omega t\partial_t+\frac{x}{2}\cos 2\omega t\partial_x-\frac{1}{4}\cos 2\omega t(u_1\partial_{u_1}+u_2\partial_{u_2})\\
& - && \frac{M\omega x^2}{2}\sin 2\omega t(\partial_{\nu_1}+\partial_{\nu_2})+\frac{1}{4}\sin 2\omega t(\partial_{\nu_1}-\partial_{\nu_2}),
\end{aligned}
$$

$$
\begin{aligned}
X_2 &= -\frac{1}{2\omega}\cos 2\omega t\partial_t + \frac{x}{2}\sin 2\omega t\partial_x - \frac{1}{4}\sin 2\omega t(u_1\partial_{u_1} + u_2\partial_{u_2}) \\
&+ \frac{M\omega x^2}{2}\cos 2\omega t(\partial_{\nu_1} + \partial_{\nu_2}) - \frac{1}{4}\cos 2\omega t(\partial_{\nu_1} - \partial_{\nu_2}), \\
X_3 &= \partial_t + \frac{\omega}{2}(\partial_{\nu_1} - \partial_{\nu_2}), \\
X_4 &= \cos\omega t\partial_x - M\omega x\sin\omega t(\partial_{\nu_1} + \partial_{\nu_2}), \\
X_5 &= \sin\omega t\partial_x + M\omega x\cos\omega t(\partial_{\nu_1} + \partial_{\nu_2}), \\
X_6 &= (\partial_{\nu_1} + \partial_{\nu_2}), \\
X_7 &= \cos(\omega t + \nu_2 - \nu_1)u_2\partial_{u_1} + \frac{u_2}{u_1}\sin(\omega t + \nu_2 - \nu_1)\partial_{\nu_1}, \\
X_8 &= \cos(\omega t + \nu_2 - \nu_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\sin(\omega t + \nu_2 - \nu_1)\partial_{\nu_2}, \\
X_9 &= u_2\partial_{u_2} - u_1\partial_{u_1}, \\
X_{10} &= \sin(\omega t + \nu_2 - \nu_1)u_2\partial_{u_1} - \frac{u_2}{u_1}\cos(\omega t + \nu_2 - \nu_1)\partial_{\nu_1}, \\
X_{11} &= -\sin(\omega t + \nu_2 - \nu_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\cos(\omega t + \nu_2 - \nu_1)\partial_{\nu_2}, \\
X_{12} &= \partial_{\nu_1} - \partial_{\nu_2}, \\
X_{13} &= u_1\partial_{u_1} + u_2\partial_{u_2}.
\end{aligned}
$$

Table 1 shows the commutation relations between the generators X_j, $j = 1, 2, \ldots, 6$. They form a Lie algebra isomorphic to $sl(2,\mathbb{R})\dot{+}h(2) = \{X_1, X_2, X_3\}\dot{+}\{X_4, X_5, X_6\}$. Table 2 shows the commutation relations between the generators X_j, $j = 7, \ldots, 12$, which form a Lie algebra isomorphic to the complex extension of $su(2)$ denoted by $su(2)^{\mathbb{C}}$. The generator X_{13} is a central element in this complete algebra. Since, the generators of table 1 commute with those of table 2, we get the symmetry Lie algebra of the set (15) as $\{sl(2,\mathbb{R})\dot{+}h(2)\} \oplus su(2)^{\mathbb{C}} \oplus \{X_{13}\}$. The interpretation of these symmetries with respect to other approaches require to compute the finite symmetry transformations of the independent and dependent variables and also a specific realization of the preceding generators. That's what we propose to do in the following subsection.

2.2 Integration of vector fields and realization of the generators

Once we integrate the vector fields, we get the one parameter groups of transformations which leave the equation (15) invariant. To the generator X_1, it corresponds the following

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$\frac{1}{2\omega}X_3$	$2\omega X_2$	$-\frac{1}{2}X_4$	$\frac{1}{2}X_5$	0
X_2	$-\frac{1}{2\omega}X_3$	0	$-2\omega X_1$	$-\frac{1}{2}X_5$	$-\frac{1}{2}X_4$	0
X_3	$-2\omega X_2$	$2\omega X_1$	0	$-\omega X_5$	ωX_4	0
X_4	$\frac{1}{2}X_4$	$\frac{1}{2}X_5$	ωX_5	0	$M\omega X_6$	0
X_5	$-\frac{1}{2}X_5$	$\frac{1}{2}X_4$	$-\omega X_4$	$-M\omega X_6$	0	0
X_6	0	0	0	0	0	0

Table 1: Commutation relations of a $sl(2,\mathbb{R}) \dot{+} h(2)$ algebra.

	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
X_7	0	X_9	$-2X_7$	0	X_{12}	$-2X_{10}$
X_8	$-X_9$	0	$2X_8$	$-X_{12}$	0	$2X_{11}$
X_9	$2X_7$	$-2X_8$	0	$2X_{10}$	$-2X_{11}$	0
X_{10}	0	X_{12}	$-2X_{10}$	0	$-X_9$	$2X_7$
X_{11}	$-X_{12}$	0	$2X_{11}$	X_9	0	$-2X_8$
X_{12}	$2X_{10}$	$-2X_{11}$	0	$-2X_7$	$2X_8$	0

Table 2: Commutation relations of a complex extension of $su(2)$.

transformation (with the integration parameter λ_1) on time and space coordinates

$$\tilde{t} = \frac{1}{\omega}\arctan\left(e^{\lambda_1}\tan\omega t\right), \quad \tilde{x} = e^{(\lambda_1/2)}x\left(\frac{1+\tan^2\omega t}{1+e^{2\lambda_1}\tan^2\omega t}\right)^{1/2} \tag{36}$$

and on the wave function

$$\begin{aligned}\tilde{\Psi}(\tilde{t},\tilde{x}) &= e^{\frac{\lambda_1}{4}}\left(\frac{1+\tan^2\omega\tilde{t}}{1+e^{-2\lambda_1}\tan^2\omega\tilde{t}}\right)^{1/4} \\ &\quad \exp\left[i\frac{M\omega\tilde{x}^2}{2\tan\omega\tilde{t}}\left(1-\frac{1+\tan^2\omega\tilde{t}}{1+e^{-2\lambda_1}\tan^2\omega\tilde{t}}\right)\right] \\ &\quad \begin{pmatrix} e^{i\omega(\tilde{t}-t)} & 0 \\ 0 & e^{-i\omega(\tilde{t}-t)}\end{pmatrix}\Psi(t,x),\end{aligned} \tag{37}$$

where t and x in the expression of $\Psi(t,x)$ given before have to be evaluated using the inverse of (36). To the generator X_2, corresponds

$$\tilde{t} = \frac{1}{\omega}\arctan\left(e^{-\lambda_2}\tan(\frac{\pi}{4}+\omega t)\right) - \frac{\pi}{4\omega}, \qquad \tilde{x} = e^{-\frac{\lambda_2}{2}}x\left(\frac{1+\tan^2(\frac{\pi}{4}+\omega t)}{1+e^{-2\lambda_2}\tan^2(\frac{\pi}{4}+\omega t)}\right)^{1/2} \tag{38}$$

and

$$\tilde{\Psi}(\tilde{t},\tilde{x}) = e^{\lambda_2/4}\left(\frac{1+\tan^2(\frac{\pi}{4}+\omega\tilde{t})}{1+e^{2\lambda_2}\tan^2(\frac{\pi}{4}+\omega\tilde{t})}\right)^{1/4}$$

$$
\times \exp\left[i\frac{M\omega\tilde{x}^2}{2\tan(\frac{\pi}{4}+\omega\tilde{t})}\left(1-\frac{1+\tan^2(\frac{\pi}{4}+\omega\tilde{t})}{1+e^{2\lambda_2}\tan^2(\frac{\pi}{4}+\omega\tilde{t})}\right)\right]
$$

$$
\times \begin{pmatrix} e^{i\omega(\tilde{t}-t)} & 0 \\ 0 & e^{-i\omega(\tilde{t}-t)} \end{pmatrix} \Psi(t,x), \tag{39}
$$

where t and x in the expression of $\Psi(t,x)$ in this equation have to be evaluated using the inverse of (38). The generator X_3 corresponds to a time translation and the following transformation of the wave function

$$
\tilde{\Psi}(\tilde{t},\tilde{x}) = \begin{pmatrix} e^{i\omega\lambda_3/2} & 0 \\ 0 & e^{-i\omega\lambda_3/2} \end{pmatrix} \Psi(\tilde{t}-\lambda_3,\tilde{x}). \tag{40}
$$

To the generator X_4, we associate the transformation

$$
\tilde{t} = t, \qquad \tilde{x} = x + \lambda_4 \cos\omega t, \tag{41}
$$

and

$$
\tilde{\Psi}(\tilde{t},\tilde{x}) = \exp\left[-iM\omega\left(\lambda_4\tilde{x}-\frac{\lambda_4^2}{2}\cos\omega\tilde{t}\right)\sin\omega\tilde{t}\right]\Psi(\tilde{t},\tilde{x}-\lambda_4\cos\omega\tilde{t}). \tag{42}
$$

The transformation associated to the generator X_5 is similar and gives

$$
\tilde{t} = t, \qquad \tilde{x} = x + \lambda_5 \sin\omega t, \tag{43}
$$

together with

$$
\tilde{\Psi}(\tilde{t},\tilde{x}) = \exp\left[iM\omega\left(\lambda_5\tilde{x}-\frac{\lambda_5^2}{2}\sin\omega\tilde{t}\right)\cos\omega\tilde{t}\right]\Psi(\tilde{t},\tilde{x}-\lambda_5\sin\omega\tilde{t}). \tag{44}
$$

The generators $X_j,\ j=6,7,\dots,13$ are not associated with space-time transformations but with transformations of the wave function which leave invariant the original set of equations. The integration of these vector fields leads to the following transformations:

$$
\begin{aligned}
\tilde{\Psi}(t,x) &= e^{i\lambda_6}\Psi(t,x), \qquad \tilde{\Psi}(t,x) = \begin{pmatrix} 1 & \lambda_7 e^{i\omega t} \\ 0 & 1 \end{pmatrix}\Psi(t,x), && (45)\\
\tilde{\Psi}(t,x) &= \begin{pmatrix} 1 & 0 \\ \lambda_8 e^{-i\omega t} & 1 \end{pmatrix}\Psi(t,x), \qquad \tilde{\Psi}(t,x) = \begin{pmatrix} e^{-\lambda_9} & 0 \\ 0 & e^{\lambda_9} \end{pmatrix}\Psi(t,x), && (46)\\
\tilde{\Psi}(t,x) &= \begin{pmatrix} 1 & -i\lambda_{10}e^{i\omega t} \\ 0 & 1 \end{pmatrix}\Psi(t,x), \qquad \tilde{\Psi}(t,x) = \begin{pmatrix} 1 & 0 \\ -i\lambda_{11}e^{-i\omega t} & 1 \end{pmatrix}\Psi(t,x), && (47)\\
\tilde{\Psi}(t,x) &= \begin{pmatrix} e^{i\lambda_{12}} & 0 \\ 0 & e^{-i\lambda_{12}} \end{pmatrix}\Psi(t,x), \qquad \tilde{\Psi}(t,x) = e^{\lambda_{13}}\Psi(t,x). && (48)
\end{aligned}
$$

Now from these finite transformations we can find a matrix realization of the infinitesimal generators of the invariance Lie algebra. It is easy to show that we get:

$$\begin{aligned}
C_-(t) &= 2\omega(iX_1 - X_2) \\
&= e^{2i\omega t}\left((\partial_t + i\omega x\partial_x + iM\omega^2 x^2 + i\frac{\omega}{2})\sigma_0 - i\frac{\omega}{2}\sigma_3\right), \qquad (49)\\
C_+(t) &= -2\omega(iX_1 + X_2) \\
&= e^{-2i\omega t}\left((\partial_t - i\omega x\partial_x + iM\omega^2 x^2 - i\frac{\omega}{2})\sigma_0 - i\frac{\omega}{2}\sigma_3\right), \qquad (50)\\
H_0 &= iX_3 = i\sigma_0\partial_t + \frac{\omega}{2}\sigma_3, \qquad (51)\\
A_{x,-}(t) &= \frac{1}{\sqrt{2M\omega}}(X_4 + iX_5) = \frac{1}{\sqrt{2M\omega}}e^{i\omega t}(M\omega x + \partial_x)\sigma_0, \qquad (52)\\
A_{x,+}(t) &= \frac{-1}{\sqrt{2M\omega}}(X_4 - iX_5) = \frac{1}{\sqrt{2M\omega}}e^{-i\omega t}(M\omega x - \partial_x)\sigma_0, \qquad (53)\\
I &= iX_6 = X_{13} = \sigma_0, \qquad (54)\\
T_+(t) &= X_7 = -iX_{10} = e^{i\omega t}\sigma_+, \qquad (55)\\
T_-(t) &= X_8 = -iX_{11} = e^{-i\omega t}\sigma_-, \qquad (56)\\
2Y &= X_9 = -iX_{12} = \sigma_3, \qquad (57)
\end{aligned}$$

where $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_1 + i\sigma_2)$ and $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_1 - i\sigma_2)$, with $\sigma_1, \sigma_2, \sigma_3$ the standard Pauli matrices. The generators $C_-,\ C_+,\ H_0,\ A_{x,-}(t),\ A_{x,+}(t)$ and I correspond exactly to the maximal kinematical algebra $sl(2,\mathbb{R}) \dot{+} h(2)$. The generators $T_+,\ T_-$ and Y correspond to the algebra $su(2)$ and are associated with the fermionic symmetries of the SUSY harmonic oscillator.

Since we expect for the SUSY harmonic oscillator the presence of bosonic (even) and fermionic (odd) symmetries, we can associate to these generators a parity, i.e., those represented in terms of diagonal matrices are called even and those represented by anti-diagonal matrices are called odd. So they close now a Lie superalgebra

$$(sl(2, \mathbb{R} \dot{+} sh(2/2)) \oplus \{Y\}.$$

The Lie superalgebra $sh(2/2)$ is given by the set $\{A_{x,-}(t),\ A_{x,+}(t),\ I;\ T_+(t), T_-(t)\}$ and the associated non-zero super-commutation relations are

$$[A_{x,-}(t), A_{x,+}(t)] = \{T_-(t), T_+(t)\} = I. \qquad (58)$$

The SUSY generators are not obtained by these procedure. Since, they play the role to exchange bosonic and fermionic fields, they are know to be associated to a composition of even and odd generators. Indeed, they may be written as the products:

$$Q_+ = \sqrt{\omega}\, T_+(t) A_{x,+}(t), \qquad Q_- = \sqrt{\omega}\, T_-(t) A_{x,-}(t), \tag{59}$$

$$S_+(t) = \sqrt{\omega}\, T_+(t) A_{x,-}(t), \qquad S_-(t) = \sqrt{\omega}\, T_-(t) A_{x,+}(t) \tag{60}$$

and they close together with the original ones, given in Eqs. (49-57), the superalgebra $osp(2/2)\dot{+}sh(2/2)$. So the prolongation method, using such matrix realization has given bosonic and fermionic symmetries which close a superalgebra. The SUSY generators present have been included by hand by taking suitable products of some basic even ($A_{x,\pm}(t)$) and odd ($T_\pm(t)$) generators. To explain why these products appear, it is convenient to relate the prolongation method for searching symmetries to the general concept of symmetry of a quantum system.

We consider the general transformation

$$\tilde{\Psi}(t,x) = X\Psi(t,x) \tag{61}$$

on the wave function $\Psi(t,x)$ of our quantum system (12), where X is a operator such that

$$(i\partial_t - H_{SUSY})\, X\Psi(t,x) = 0, \tag{62}$$

i.e., X transforms solutions of our system into solutions. The operator X of (62) is called a symmetry operator of the model under study. In this more general context it is clear that, if two operators X and Y satisfy (62), the product XY also. Moreover, if eq.(12) satisfies the superposition principle of solutions, the linear combination $\alpha X + \beta Y$, where $\alpha, \beta \in \mathbb{C}$ satisfies also (62). But the complete set of operators obtained by this procedure does not necessarily close a Lie algebra or superalgebra.

Let us take the operator X in (62) to be on the differential form[6]

$$X = \sum_{\mu=0}^{4} \left[\sigma_\mu \left(\phi_\mu^0(t,x) + \phi_\mu^1(t,x)\partial_t + \phi_\mu^2(t,x)\partial_x\right)\right], \tag{63}$$

where the $\phi_\mu^k(t,x)$, $k = 0,1,2; \mu = 1,\ldots,4$, are real functions of t and x. Comparing the coefficients of several independent products of derivatives we get a system of PDE's which

can be solved to determine $\phi^0_\mu(t,x), \phi^1_\mu(t,x)$ and $\phi^2_\mu(t,x)$, up to a finite number of arbitrary integration constants. Solving this system, inserting the results in (63), identifying the different operators according to each integration constants, and finally, taking suitable combinations of these operators, we obtain the generators (49-57) and also other ones. These last are the second order products of the original ones, i.e.,

$$YC_+(t), \quad YC_-(t), \quad YA_{x,+}(t), \quad YA_{x,-}(t), \quad YX_3, \tag{64}$$

$$T_+(t)C_+(t), \quad T_+(t)C_-(t), \quad T_+(t)A_{x,+}(t), \quad T_+A_{x,-}(t), \quad T_+(t)X_3, \tag{65}$$

$$T_-(t)C_+(t), \quad T_-(t)C_-(t), \quad T_-(t)A_{x,+}(t), \quad T_-A_{x,-}(t), \quad T_-(t)X_3. \tag{66}$$

It is easy to show that the all set of symmetries does not close a Lie algebra or a Lie superalgebra and the only way to close the structure under both commutation and anti-commutation relations is to select among all the preceding products the four ones given in (59-60).

Let us finally mention that on the space of solutions of the equation (12), the superalgebra $osp(2/2) \dotplus sh(2/2)$ may be expressed as

$$\begin{aligned}
C_-(t) &= i\omega e^{2i\omega t}\frac{a_x^2}{2}\sigma_0, \quad C_+(t) = i\omega e^{-2i\omega t}\frac{(a_x^\dagger)^2}{2}\sigma_0, \quad H_0 = \omega\left(a_x^\dagger a_x + \frac{1}{2}\right), && (67)\\
A_{x,-}(t) &= e^{i\omega t}a_x\sigma_0, \qquad A_{x,+}(t) = e^{-i\omega t}a_x^\dagger\sigma_0, \qquad I = \sigma_0, && (68)\\
T_+(t) &= e^{i\omega t}\sigma_+, \qquad T_-(t) = e^{-i\omega t}\sigma_-, \qquad Y = \frac{\sigma_3}{2}, && (69)\\
Q_+ &= \sqrt{\omega}\ a_x^\dagger\sigma_+, \qquad Q_- = \sqrt{\omega}\ a_x\sigma_-, && (70)\\
S_+(t) &= \sqrt{\omega}\ e^{2i\omega t}a_x\sigma_+, \qquad S_-(t) = \sqrt{\omega}\ e^{-2i\omega t}a_x^\dagger\sigma_-, && (71)
\end{aligned}$$

where

$$a_x = \frac{1}{\sqrt{2M\omega}}(M\omega x + \partial_x), \quad a_x^\dagger = \frac{1}{\sqrt{2M\omega}}(M\omega x - \partial_x), \tag{72}$$

are the usual annihilation and creation operators, respectively. They satisfy the commutation relation

$$[a_x, a_x^\dagger] = 1. \tag{73}$$

Table 3 shows the commutation and anticommutation relations between the generators of the orthosymplectic superalgebra and, as expected, the generators $Q_\pm$ are the supercharges of the system. This means that they satisfy

$$\{Q_+, Q_-\} = H_0 - \omega Y = H_{SUSY}, \quad (Q_\pm)^2 = 0 \tag{74}$$

	H_0	$C_-(t)$	$C_+(t)$	Y	Q_-	Q_+	$S_-(t)$	$S_+(t)$
H_0	0	$-2\omega C_-$	$2\omega C_+$	0	$-\omega Q_-$	ωQ_+	ωS_-	$-\omega S_+$
$C_-(t)$	$2\omega C_-(t)$	0	$-\omega H_0$	0	0	$i\omega S_+$	$i\omega Q_-$	0
$C_+(t)$	$-2\omega C_+$	ωH_0	0	0	$-i\omega S_-$	0	0	$-i\omega Q_+$
Y	0	0	0	0	$-Q_-$	Q_+	$-S_-$	S_+
Q_-	ωQ_-	0	$i\omega S_-$	Q_-	0	$H_0 - \omega Y$	0	$-2iC_-$
Q_+	ωQ_+	$i\omega S_+$	0	$-Q_+$	$H_0 - \omega Y$	0	$-2iC_+$	0
$S_-(t)$	$-\omega S_-$	$-i\omega Q_-$	0	$2S_-$	0	$-2iC_+$	0	$H_0 + \omega Y$
$S_+(t)$	ωS_+	0	$i\omega Q_+$	$-2S_-$	$-2iC_-$	0	$H_0 + \omega Y$	0

Table 3: Super-commutation relations of a $osp(2/2)$ superalgebra.

	H_0	$C_-(t)$	$C_+(t)$	Y	Q_-	Q_+	$S_-(t)$	$S_+(t)$
$A_{x,-}(t)$	$\omega A_{x,-}$	0	$i\omega A_{x,+}$	0	0	$\sqrt{\omega}T_+$	$\sqrt{\omega}T_-$	0
$A_{x,+}(t)$	$-\omega A_{x,+}$	$-i\omega A_{x,-}$	0	0	$-\sqrt{\omega}T_-$	0	0	$-\sqrt{\omega}T_+$
I	0	0	0	0	0	0	0	0
$T_-(t)$	0	0	0	T_-	0	$\sqrt{\omega}A_{x,+}$	0	$\sqrt{\omega}A_{x,-}$
$T_+(t)$	0	0	0	$-T_+$	$\sqrt{\omega}A_{x,-}$	0	$\sqrt{\omega}A_{x,+}$	0

Table 4: Super-commutation relations between the generators of $osp(2/2)$ and $sh(2/2)$.

and

$$[H_{SUSY}, Q_\pm] = 0. \tag{75}$$

Table 4 shows the structure relations between the generators of $osp(2/2)$ and $sh(2/2)$.

3 A non-relativistic spin-$\frac{1}{2}$ particle in a constant magnetic field

A problem which is related to the preceding one is the search for symmetries and supersymmetries of the Schrödinger-Pauli equation describing the motion in the plane of a non-relativistic spin-$\frac{1}{2}$ particle of electric charge e in a constant magnetic field $\vec{B} = (0, 0, B)$ orthogonal to the plane. We thus have the equation

$$(i\partial_t - H_P)\, \Psi(t, x, y) = 0, \tag{76}$$

where $\Psi(t, x, y)$ is as given in (14) except that ψ_1 and ψ_2 depend on t, x and y. The Hamiltonian is explicitly given by

$$H_P = \frac{\left(\vec{\sigma} \cdot (\vec{p} - e\vec{A})\right)^2}{2M} = \frac{(\vec{p} - e\vec{A})^2}{2M} + i\vec{\sigma} \cdot ((\vec{p} - e\vec{A}) \times (\vec{p} - e\vec{A})). \tag{77}$$

The vector $\vec{\sigma}$ is given by $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_i's are the usual Pauli matrices, $\vec{p} = (p_x, p_y, 0)$ is the linear momentum and

$$\vec{A} = (-\frac{1}{2}By\,,\;\frac{1}{2}Bx\,,\;0) \tag{78}$$

is the vector potential in the symmetric gauge . We can thus write the equation (76) as the following set of equations:

$$\left\{i\partial_t + \frac{1}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - ieB\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{e^2B^2}{4}(x^2+y^2) + eB_\alpha\right)\right\}\psi_\alpha(t,x,y) = 0, \tag{79}$$

with $\alpha = 1, 2$ and where we have set $B_1 = B$, $B_2 = -B$.

To get the infinitesimal generators corresponding to the symmetries of this set, we can again apply the prolongation method where the wave function $\Psi(t, x, y)$ may be written now as

$$\psi_1(t,x,y) = u_1(t,x,y)e^{i\nu_1(t,x,y)}, \qquad \psi_2(t,x,y) = u_2(t,x,y)e^{i\nu_2(t,x,y)}, \tag{80}$$

where u_1, u_2, ν_1 et ν_2 are real functions. The corresponding vector field is

$$v = \xi_1\partial_t + \xi_2\partial_x + \xi_3\partial_y + \phi_1\partial_{u_1} + \phi_2\partial_{u_2} + \varphi_1\partial_{\nu_1} + \varphi_2\partial_{\nu_2}, \tag{81}$$

where ξ_j $(j = 1, 2, 3)$, ϕ_α and φ_α $(\alpha = 1, 2)$ are real functions of $t, x, y, u_1, u_2, \nu_1, \nu_2$. As before, it is easier to make the calculation using the complex form of the vector field

$$v = \xi_1\partial_t + \xi_2\partial_x + \xi_3\partial_y + \Phi_1\partial_{\psi_1} + \bar{\Phi}_1\partial_{\bar{\psi}_1} + \Phi_2\partial_{\psi_2} + \bar{\Phi}_2\partial_{\bar{\psi}_2}, \tag{82}$$

and after to go back to the real form. We finally get the following generators, where we have introduced $\omega = \frac{eB}{2M}$:

$$\begin{aligned}
X_0 &= \partial_t - \omega(x\partial_y - y\partial_x) + \omega(\partial_{\nu_1} - \partial_{\nu_2}), \\
X_1 &= \cos 2\omega t\partial_t - \omega(x\sin 2\omega t - y\cos 2\omega t)\partial_x - \omega(x\cos 2\omega t + y\sin 2\omega t)\partial_y \\
&- M\omega^2(x^2+y^2)\cos 2\omega t(\partial_{\nu_1} + \partial_{\nu_2}) + \omega(\sin 2\omega t\;(u_1\partial_{u_1} + u_2\partial u_2) \\
&+ \cos 2\omega t\;(\partial_{\nu_1} - \partial_{\nu_2})), \\
X_2 &= -\sin 2\omega t\partial_t - \omega(x\cos 2\omega t + y\sin 2\omega t)\partial_x + \omega(x\sin 2\omega t - y\cos 2\omega t)\partial_y \\
&+ M\omega^2(x^2+y^2)\sin 2\omega t(\partial_{\nu_1} + \partial_{\nu_2}) + \omega(\cos 2\omega t(u_1\partial_{u_1} + u_2\partial u_2) \\
&- \sin 2\omega t(\partial_{\nu_1} - \partial_{\nu_2})),
\end{aligned}$$

	X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_0	0	$2X_2$	$-2\omega X_1$	0	ωX_5	$-\omega X_4$	0	ωX_8	$-\omega X_7$
X_1	$-2\omega X_2$	0	$-2\omega X_0$	0	$\frac{1}{2}X_8$	$\frac{1}{2}X_7$	0	$2\omega^2 X_5$	$2\omega^2 X_4$
X_2	$2\omega X_1$	$2\omega X_0$	0	0	$-\frac{1}{2}X_7$	$\frac{1}{2}X_8$	0	$-2\omega^2 X_4$	$2\omega^2 X_5$
X_3	0	0	0	0	X_5	$-X_4$	0	$-X_8$	X_7
X_4	$-\omega X_5$	$-\frac{1}{2}X_8$	$\frac{1}{2}X_7$	$-X_5$	0	$-\frac{M}{2\omega}X_6$	0	0	0
X_5	ωX_4	$-\frac{1}{2}X_7$	$-\frac{1}{2}X_8$	X_4	$\frac{M}{2\omega}X_6$	0	0	0	0
X_6	0	0	0	0	0	0	0	0	0
X_7	$-\omega X_8$	$-2\omega^2 X_5$	$2\omega^2 X_4$	X_8	0	0	0	0	$-2M\omega X_6$
X_8	ωX_7	$-2\omega^2 X_4$	$-2\omega^2 X_5$	$-X_7$	0	0	0	$2M\omega X_6$	0

Table 5: Commutation relations for a $\{sl(2,\mathbb{R})\oplus so(2)\}\dot{+}h(4)$ algebra.

$$
\begin{aligned}
X_3 &= x\partial_y - y\partial_x, \\
X_4 &= -\frac{1}{2\omega}(\cos 2\omega t\partial_x - \sin 2\omega t\partial_y) + \frac{M}{2}(x\sin 2\omega t + y\cos 2\omega t)(\partial_{\nu_1}+\partial_{\nu_2}), \\
X_5 &= \frac{1}{2\omega}(\sin 2\omega t\partial_x + \cos 2\omega t\partial_y) + \frac{M}{2}(x\cos 2\omega t - y\sin 2\omega t)(\partial_{\nu_1}+\partial_{\nu_2}), \\
X_6 &= (\partial_{\nu_1}+\partial_{\nu_2}), \\
X_7 &= \partial_x + M\omega y(\partial_{\nu_1}+\partial_{\nu_2}), \\
X_8 &= \partial_y - M\omega x(\partial_{\nu_1}+\partial_{\nu_2}), \\
X_9 &= \cos(2\omega t+\nu_2-\nu_1)u_2\partial_{u_1} + \frac{u_2}{u_1}\sin(2\omega t+\nu_2-\nu_1)\partial_{\nu_1}, \\
X_{10} &= \cos(2\omega t+\nu_2-\nu_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\sin(2\omega t+\nu_2-\nu_1)\partial_{\nu_2}, \\
X_{11} &= u_2\partial_{u_2} - u_1\partial_{u_1} \\
X_{12} &= \sin(2\omega t+\nu_2-\nu_1)u_2\partial_{u_1} - \frac{u_2}{u_1}\cos(2\omega t+\nu_2-\nu_1)\partial_{\nu_1}, \\
X_{13} &= -\sin(2\omega t+\nu_2-\nu_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\cos(2\omega t+\nu_2-\nu_1)\partial_{\nu_1}, \\
X_{14} &= \partial_{\nu_1} - \partial_{\nu_2}, \\
X_{15} &= u_1\partial_{u_1} + u_2\partial_{u_2},
\end{aligned}
$$

The commutation relations between the generators X_μ ($\mu = 0,1,2,\ldots,8$) are given in table 5 and give an algebra isomporphic to $\{sl(2,\mathbb{R})\oplus so(2)\}\dot{+}h(4)$. It is easy to show that the generators X_μ ($\mu = 9,10,\ldots,14$) form an algebra isomorphic to $su(2)^{\mathbb{C}}$ and equivalent to the one given in table 2 for the SUSY harmonic oscillator. These two sets commute with each other and X_{15} is a central element. So we have an algebra isomorphic to $\{(sl(2,\mathbb{R})\oplus so(2))\dot{+}\, h(4)\}\oplus su(2)^{\mathbb{C}}\oplus\{X_{15}\}$.

Once again, we can get a specific matrix realization proceeding as in the case of the SUSY harmonic oscillator. It is easy to show that we have the following form for the generators of symmetries of the equation (76):

$$
\begin{aligned}
H_0 &= iX_0 = (i\partial_t + \omega(xp_y - yp_x))\,\sigma_0 + \omega\sigma_3, && (83)\\
C_-(t) &= \frac{(X_1 - iX_2)}{2} = \frac{e^{2i\omega t}}{2}\{(\partial_t - i\omega(xp_y - yp_x) \\
&- \omega(xp_x + yp_y) + i\omega + iM\omega^2(x^2+y^2))\sigma_0 - i\omega\sigma_3\}, && (84)\\
C_+(t) &= \frac{(X_1 + iX_2)}{2} = \frac{e^{-2i\omega t}}{2}\{(\partial_t - i\omega(xp_y - yp_x) \\
&+ \omega(xp_x + yp_y) - i\omega + iM\omega^2(x^2+y^2))\sigma_0 - i\omega\sigma_3\}, && (85)\\
L &= -iX_3 = xp_y - yp_x, && (86)\\
\mathcal{A}(t) &= \sqrt{\frac{\omega}{M}}(iX_4 + X_5) = \frac{e^{2i\omega t}}{2\sqrt{\omega M}}[(p_x + ip_y) - iM\omega(x+iy)]\sigma_0, && (87)\\
\mathcal{A}^\dagger(t) &= \sqrt{\frac{\omega}{M}}(iX_4 - X_5) = \frac{e^{-2i\omega t}}{2\sqrt{\omega M}}[(p_x - ip_y) + iM\omega(x-iy)]\sigma_0, && (88)\\
I &= X_6 = X_{15} = \sigma_0, && (89)\\
A_- &= \frac{-1}{2\sqrt{M\omega}}(X_8 + iX_7) = \frac{1}{2\sqrt{M\omega}}[(p_x - ip_y) - iM\omega(x-iy)]\sigma_0, && (90)\\
A_+ &= \frac{1}{2\sqrt{M\omega}}(X_8 - iX_7) = \frac{1}{2\sqrt{M\omega}}[(p_x + ip_y) + iM\omega(x+iy)]\sigma_0, && (91)\\
T_+(t) &= X_9 = -iX_{12} = e^{2i\omega t}\sigma_+, \qquad T_-(t) = X_{10} = -iX_{13} = e^{-2i\omega t}\sigma_-, && (92)\\
Y &= X_{11} = -iX_{14} = \sigma_3. && (93)
\end{aligned}
$$

We see that H_0 is essentially the Hamiltonian of the harmonic oscillator in two dimensions, $C_\pm(t)$ correspond to the so-called conformal transformations and L is the angular momentum. The two sets $\{\mathcal{A}(t),\ \mathcal{A}^\dagger(t)\}$ and $\{A_-,\ A_+\}$ may be associated to pairs of annihilation and creation operators and I represents the identity generator. Indeed, they can be written as

$$\mathcal{A}(t) = \frac{1}{\sqrt{2}}e^{2i\omega t}(a_y - ia_x)\sigma_0, \qquad \mathcal{A}^\dagger(t) = \frac{1}{\sqrt{2}}e^{-2i\omega t}(a_y^\dagger + ia_x^\dagger)\sigma_0, \tag{94}$$

$$A_- = -\frac{1}{\sqrt{2}}(a_y + ia_x)\sigma_0, \qquad A_+ = -\frac{1}{\sqrt{2}}(a_y^\dagger - ia_x^\dagger)\sigma_0, \tag{95}$$

where $a_x,\ a_x^\dagger,\ a_y$ and $a_y^\dagger$ are defined as in (72). The set $\{T_+(t),\, T_-(t), Y\}$ corresponds to the Lie algebra $su(2)$. These generators form the maximal kinematical algebra of the Pauli equation (76) which is $\{(sl(2,\mathbb{R}) \oplus so(2)) \dotplus h(4)\} \oplus su(2)$.

Now the products of the generators which will lead to the invariance superalgebras are obtained from the sets $\{\mathcal{A}(t),\ \mathcal{A}^{\dagger}(t)\}$, $\{A_-,\ A_+\}$ and $\{T_+(t),\ T_-(t)\}$. There are eight possible products. If we first take

$$Q_- = \sqrt{2\omega}\ \mathcal{A}(t)\, T_-(t), \qquad Q_+ = \sqrt{2\omega}\ \mathcal{A}^{\dagger}(t)\, T_+(t), \tag{96}$$

we see that they are independent on time as it was the case with the SUSY harmonic oscillator. Moreover they satisfy

$$Q_-^2 = Q_+^2 = 0 \tag{97}$$

and

$$\{Q_-, Q_+\} = H_P = H_0 - \omega L - \omega Y, \qquad [H_P, Q_\mp] = 0. \tag{98}$$

This means that $Q_\mp$ are the supercharges for the Pauli Hamiltonian H_P and a class of supersymmetries of our system. Another set of supersymmetries is found to be

$$S_-(t) = \sqrt{2\omega}\ A_+\ T_-(t), \qquad S_+(t) = \sqrt{2\omega}\ A_-\ T_+(t). \tag{99}$$

Let us insist on the fact that the two sets of annihilation and creation operators $\{\mathcal{A}(t),$ $\mathcal{A}^{\dagger}(t)\}$ and $\{A_-,\ A_+\}$ thus appear in these supersymmetry generators. The operators $S_\pm(t)$, which are now time dependent, satisfy

$$(S_-(t))^2 = (S_+(t))^2 = 0 \tag{100}$$

and

$$\{S_-(t), S_+(t)\} = H_0 + \omega L + \omega Y, \qquad [i\partial_t - H_P, S_\pm(t)] = 0. \tag{101}$$

The other structure relations are computed and the non-zero ones are

$$[H_0, Q_\pm] = \pm\omega Q_\pm \quad , \qquad [H_0, S_\pm] = \mp\omega S_\pm(t), \tag{102}$$

$$[C_+(t), Q_-] = i\omega S_-(t) \quad , \qquad [C_-(t), Q_+] = -i\omega S_+(t), \tag{103}$$

$$[Y, Q_\pm] = \pm 2Q_\pm \quad , \qquad [Y, S_\pm(t)] = \pm 2S_\pm(t), \tag{104}$$

$$\{Q_-, S_+(t)\} = 2iC_-(t) \quad , \qquad \{Q_+, S_-\} = 2iC_+(t). \tag{105}$$

This means that the generators $\{H_0,\ C_\pm(t),\ Y,\ Q_\pm,\ S_\pm(t)\}$ close the orthosymplectic superalgebra $osp(2/2)$. Together with the other generators of the maximal kinematical algebra, we get the so-called maximal kinematical superalgebra of the Pauli equation

defined as $\{osp(2/2)\oplus so(2)\} \dot{+} sh(2/4)$, where $sh(2/4)$ is the Heisenberg-Weyl superalgebra generated by the fermionic generators $T_{\pm}(t)$, the bosonic generators $\mathcal{A}(t), \mathcal{A}^{\dagger}(t), A_{-}, A_{+}$ and the identity I [7]. Let us finally mention that the remaining products of $\{\mathcal{A}(t),\ \mathcal{A}^{\dagger}(t)\}$ and $\{A_{-},\ A_{+}\}$ with $\{T_{+}(t),\ T_{-}(t)\}$ give rise to the following generators

$$U_{+}(t) = \sqrt{2\omega} A_{+}\, T_{+}(t), \qquad U_{-}(t) = \sqrt{2\omega} A_{-}\, T_{-}(t), \tag{106}$$

$$V_{+}(t) = \sqrt{2\omega} \mathcal{A}(t)\, T_{+}(t), \qquad V_{-}(t) = \sqrt{2\omega} \mathcal{A}^{\dagger}(t)\, T_{-}(t). \tag{107}$$

They have been introduced in [7] and are contained in the maximal dynamical superalgebra of the system under consideration which is $osp(2/4) \ni sh(2/4)$. Indeed to close the structure with these additional supersymmetries, it is necessary to include new even generators of the dynamical algebra of our model.

4 The Jaynes-Cummings model

Now we consider the system described by a particle of electric charge e, spin $\frac{1}{2}$ and mass M moving in the plane in the presence of constant electric $\vec{E}$ and magnetic $\vec{B}$ fields which are both perpendicular to the plane. It has been shown to be related to the Jaynes-Cummings model [19]. Indeed the Hamiltonian characterising such a system is given by

$$H = \frac{\left(\vec{p} - e\vec{A}\right)^2}{2M} - \frac{e}{2M}\vec{B}\cdot\vec{\sigma} + \frac{e}{4M^2}\vec{E}\cdot\left(\vec{\sigma}\times(\vec{p} - e\vec{A})\right) + e\vec{E}\cdot\vec{x}, \tag{108}$$

where $\vec{x}$ is the position vector of the particle and $\vec{A}$ is the potential vector given in (78). We are again interested in the motion in the xy-plane for which the contribution of the preceding Hamiltonian is

$$\begin{aligned} H_{JC} &= \frac{1}{2M}(p_x^2 + p_y^2 + \frac{e^2B^2}{4}(x^2+y^2) + eB(yp_x - xp_y))\sigma_0 - \frac{eB}{2M}\sigma_3 \qquad (109)\\ &+ \frac{ieE}{4M^2}\left((p_x - ip_y) + i\frac{eB}{2}(x - iy)\right)\sigma_{+} - \frac{ieE}{4M^2}\left((p_x + ip_y) - i\frac{eB}{2}(x + iy)\right)\sigma_{-}, \end{aligned}$$

or again,

$$H_{JC} = H_P + \frac{ieE}{4M^2}\left((p_x - ip_y) + i\frac{eB}{2}(x - iy)\right)\sigma_{+} - \frac{ieE}{4M^2}\left((p_x + ip_y) - i\frac{eB}{2}(x + iy)\right)\sigma_{-}, \tag{110}$$

where H_P is the Pauli Hamiltonian (77). Let us assume without loss of generality that $e > 0$ and introduce the operators

$$\mathcal{A} = \mathcal{A}(0) = \frac{1}{\sqrt{2eB}}\Big((p_x + ip_y) - i\frac{eB}{2}(x+iy)\Big)\sigma_0, \tag{111}$$

$$\mathcal{A}^\dagger = \mathcal{A}^\dagger(0) = \frac{1}{\sqrt{2eB}}\Big((p_x - ip_y) + i\frac{eB}{2}(x-iy)\Big)\sigma_0, \tag{112}$$

which satisfy the commutation relation

$$[\mathcal{A}, \mathcal{A}^\dagger] = I. \tag{113}$$

These operators are nothing else than the generators given by (87) and (88) at $t = 0$ when we take again $\omega = \frac{eB}{2M}$. Let us recall that they correspond to symmetries of the Pauli system. Here we will see that they are not symmetries of the JC model. The Hamiltonian H_{JC} can thus be written on the form

$$H_{JC} = \tilde{\omega}\left(\mathcal{A}^\dagger\mathcal{A} + \frac{1}{2}\right)\sigma_0 - \frac{\tilde{\omega}}{2}\sigma_3 + \kappa\mathcal{A}^\dagger\sigma_+ + \bar{\kappa}\mathcal{A}\sigma_-, \tag{114}$$

where $\tilde{\omega} = 2\omega = \frac{eB}{M}$ and $\kappa = \frac{ieE\sqrt{2eB}}{4M^2}$. This means that the Hamiltonian (114) is a realization of the JC Hamiltonian [8, 19, 20] in the special case where the detuning between the frequency of the cavity mode and the atom transition frequency is zero. We also see a close connection with the SUSY harmonic oscillator Hamiltonian described in terms of new annihilation and creation operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ as given in (111-112).

4.1 Lie algebra of symmetries

We are interested to determine the symmetries of the corresponding evolution equation

$$(i\partial_t - H_{JC})\Psi(t,x,y) = 0, \tag{115}$$

where $\Psi(t,x,y)$ is again a two component wave function as in the Pauli equation considered in the preceding section. It can be written explicitly

$$\begin{aligned} i\psi_{1,t} &+ \frac{1}{2M}\left[\psi_{1,xx} + \psi_{1,yy} - ieB(x\ \psi_{1,y} - y\ \psi_{1,x}) - \frac{e^2B^2}{4}(x^2+y^2)\ \psi_1 + eB_1\psi_1\right] \\ &+ \frac{eE_1}{4M^2}\left(\psi_{2,x} + i\frac{eB}{2}y\ \psi_2\right) + \frac{eE}{4M^2}\left(i\psi_{2,y} + \frac{eB}{2}x\ \psi_2\right) = 0, \end{aligned} \tag{116}$$

$$\begin{aligned} i\psi_{2,t} &+ \frac{1}{2M}\left[\psi_{2,xx} + \psi_{2,yy} - ieB(x\ \psi_{2,y} - y\ \psi_{2,x}) - \frac{e^2B^2}{4}(x^2+y^2)\ \psi_2 + eB_2\psi_2\right] \\ &+ \frac{eE_2}{4M^2}\left(\psi_{1,x} + i\frac{eB}{2}y\ \psi_1\right) + \frac{eE}{4M^2}\left(i\psi_{1,y} + \frac{eB}{2}x\ \psi_1\right) = 0, \end{aligned} \tag{117}$$

where we have set $B_1 = B$, $B_2 = -B$, $E_1 = -E$ and $E_2 = E$.

As in the preceding sections, to get the symmetries of the system (116-117), we apply the prolongation method to it and the conjugated system. Once again the vector field has the form (82) with

$$\xi_1 = \delta_1, \qquad \xi_2(t,y) = -\delta_2 y + \delta_3, \qquad \xi_3(t,x) = \delta_2 x + \delta_4, \tag{118}$$

$$\Phi_1(t,x,y) = A_0(t,x,y) + A_1(x,y)\psi_1,\ \Phi_2(t,x,y) = C_0(t,x,y) + C_2(x,y)\psi_2, \tag{119}$$

where

$$\begin{aligned} A_1(x,y) &= -i\frac{eB}{2}(\delta_4 x - \delta_3 y) - i\frac{\delta_2}{2} + (\delta_6 + i\delta_5), & (120)\\ C_2(x,y) &= -i\frac{eB}{2}(\delta_4 x - \delta_3 y) + i\frac{\delta_2}{2} + (\delta_6 + i\delta_5). & (121) \end{aligned}$$

The δ_j $(j = 1, \dots 6)$ are arbitrary real constants and $A_0(t,x,y)$ and $C_0(t,x,y)$ are arbitrary functions that satisfy the system (116-117) for $\psi_1 = A_0$ and $\psi_2 = C_0$. The finite dimensional Lie algebra of symmetries is thus formed by the following infinitesimal generators:

$$\begin{aligned} X_1 &= \partial_t,\\ X_2 &= (x\partial_y - y\partial_x) - \frac{i}{2}(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}) + \frac{i}{2}(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}),\\ X_3 &= \partial_x + i\frac{eB}{2}y\ (\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1} + \psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}),\\ X_4 &= \partial_y - i\frac{eB}{2}x\ (\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1} + \psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}),\\ X_5 &= i(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}) + i(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}),\\ X_6 &= (\psi_1\partial_{\psi_1} + \bar{\psi}_1\partial_{\bar{\psi}_1}) + (\psi_2\partial_{\psi_2} + \bar{\psi}_2\partial_{\bar{\psi}_2}). \end{aligned}$$

Using the real components of the wave function $\Psi(t,x,y)$ given by (80), the infinitesimal generators take the form

$$\begin{aligned} X_1 &= \partial_t, & (122)\\ X_2 &= (x\partial_y - y\partial_x) - \frac{1}{2}(\partial_{\nu_1} - \partial_{\nu_2}), & (123)\\ X_3 &= \partial_x + \frac{eB}{2}y(\partial_{\nu_1} + \partial_{\nu_2}), & (124)\\ X_4 &= \partial_y - \frac{eB}{2}x(\partial_{\nu_1} + \partial_{\nu_2}), & (125)\\ X_5 &= (\partial_{\nu_1} + \partial_{\nu_2}), & (126)\\ X_6 &= u_1\partial_{u_1} + u_2\partial_{u_2}. & (127) \end{aligned}$$

It is easy to see that X_1 and X_6 are both central elements. The generator X_2 corresponds to a $so(2)$ algebra and the set $\{X_3, X_4, X_5\}$ generates $h(2)$. These last generators are thus associated to a Lie algebra isomorphic to $so(2) \dotplus h(2)$ and satisfy the following non-zero commutation relations

$$[X_2, X_3] = -X_4, \qquad [X_2, X_4] = X_3, \tag{128}$$

$$[X_3, X_4] = -eBX_5. \tag{129}$$

Integration of the vector fields gives rise to finite transformations of independent and dependent variables which leave the equation (115) invariant. Explicitly, to X_1 corresponds the invariance under time translation such that:

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = \Psi(\tilde{t} - \lambda_1, \tilde{x}, \tilde{y}). \tag{130}$$

The vector field X_2 corresponds to the invariance under rotations in the xy-plane. The transformation is

$$(\tilde{t}, \tilde{x}, \tilde{y}) = (t, x\cos\lambda_2 - y\sin\lambda_2, x\sin\lambda_2 + y\cos\lambda_2) \tag{131}$$

and

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = \begin{pmatrix} e^{-\frac{i}{2}\lambda_2} & 0 \\ 0 & e^{\frac{i}{2}\lambda_2} \end{pmatrix} \Psi(\tilde{t}, \tilde{x}\cos\lambda_2 + \tilde{y}\sin\lambda_2, -\tilde{x}\sin\lambda_2 + \tilde{y}\cos\lambda_2). \tag{132}$$

The vector fields X_3 and X_4 correspond to the invariance under space translations. We have

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = e^{i\frac{eB}{2}\lambda_3\tilde{y}}\ \Psi(\tilde{t}, \tilde{x} - \lambda_3, \tilde{y}) \tag{133}$$

and

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = e^{-i\frac{eB}{2}\lambda_4\tilde{x}}\ \Psi(\tilde{t}, \tilde{x}, \tilde{y} - \lambda_4), \tag{134}$$

respectively. The vector fields X_5 and X_6 are not related to space-time transformations but to the following phase and scale transformations of the wave function respectively:

$$\tilde{\Psi}(t, x, y) = e^{i\lambda_5}\Psi(t, x, y) \tag{135}$$

and

$$\tilde{\Psi}(t, x, y) = e^{\lambda_6}\Psi(t, x, y). \tag{136}$$

From these finite transformations, we easily get a matrix realization of the Lie algebra of symmetries of the equation (115).

$$
\begin{aligned}
X_1 &= \sigma_0 \partial_t, \qquad X_2 = (x\partial_y - y\partial_x)\sigma_0 + \frac{i}{2}\sigma_3, && (137)\\
X_3 &= \left(\partial_x - i\frac{eB}{2}y\right)\sigma_0, \qquad X_4 = \left(\partial_y + i\frac{eB}{2}x\right)\sigma_0, && (138)\\
X_5 &= -iX_6 = -iI = -i\sigma_0. && (139)
\end{aligned}
$$

Now X_1, while acting on the space of solution of (115), is the Hamiltonian H_{JC} as expected and $J = -iX_2$ is the total angular momentum. Complex linear combinations of X_3 and X_4 give

$$
A_- = \frac{1}{\sqrt{2eB}}\left[(p_x - ip_y) - i\frac{eB}{2}(x - iy)\right]\sigma_0 \tag{140}
$$

and

$$
A_+ = \frac{1}{\sqrt{2eB}}\left[(p_x + ip_y) + i\frac{eB}{2}(x + iy)\right]\sigma_0 \tag{141}
$$

which satisfy

$$
[A_-, A_+] = I. \tag{142}
$$

They are exactly the symmetries of the Pauli Hamiltonian as given in (90) and (91) and close together with the identity I the algebra $h(2)$. The operators $\{H_{JC}, J, A_\mp, I\}$ are thus associated to the maximal kinematical invariance algebra of the JC model and is isomorphic to $(so(2) \dotplus h(2)) \oplus u(1)$. They are all time independent and thus commute with H_{JC}. Moreover they are all diagonal matrices and correspond necessarily to even generators.

It is neither possible from the prolongation method to produce a Lie superalgebra of symmetries nor a set of supersymmetries as it was the case for the preceding models. We know in fact [9, 10] that the standard JC model does not admit a N=2 supersymmetry. It is due to the presence in the Hamiltonian (114) of the additional term $(\kappa\mathcal{A}^\dagger\sigma_+ + \bar{\kappa}\mathcal{A}\sigma_-)$ which can be written as

$$
\mathcal{Q} = \frac{1}{\sqrt{2\omega}}(\kappa Q_+ + \bar{\kappa} Q_-) = \kappa\mathcal{A}^\dagger\sigma_+ + \bar{\kappa}\mathcal{A}\sigma_-, \tag{143}
$$

where Q_+ and Q_- are given in (96). Due to the value of κ, such a term is essentially a multiple of the combination $Q_+ - Q_-$. It is an additional symmetry of H_{JC} which can not be obtained from the prolongation method. It commutes with $A_\pm$, σ_0 and J so that

the set $\{H_{JC}, A_\pm, I, J,\ Q_+ - Q_-\}$ close a Lie algebra but not a Lie superalgebra. Indeed, $(Q_+ - Q_-)^2$ is not a linear combination of the preceding generators.

5 A generalized Jaynes-Cummings model

As already mentioned in the approach by Andreev and Lerner [9], to be able to get a supersymmetry for the JC model, it is necessary to consider a 4 by 4 matrix version of the JC Hamiltonian. Let us show here how the prolongation method may be adapted to such a model and will lead to the presence of supersymmetry generators as for the SUSY harmonic oscillator and the Pauli Hamiltonians. We first construct a generalized JC model for which the prolongation method will produce symmetries associated with a Lie superalgebra. Next we examine the possibility to get supersymmetries for a symmetrized JC model and compare the results with the ones associated with the so-called standard SUSY JC model [21].

Let us start with a version of the JC Hamiltonian of the following type:

$$H_T = \begin{pmatrix} H_{JC} - \frac{eB}{2M}\alpha\sigma_0 & 0 \\ 0 & H_{JC} - \frac{eB}{2M}\beta\sigma_0 \end{pmatrix}, \tag{144}$$

where H_{JC} is given in (109) and α, β are real parameters. The Schrödinger type equation is

$$(i\partial_t - H_T)\Psi(t, x, y) = 0, \tag{145}$$

where $\Psi(t, x, y)$ is a four component wave function whose entries are complex valued functions equal to $\psi_\rho(t, x, y)$ $(\rho = 1, \dots, 4)$. Since (144) is diagonal we get a system of four equations which are firstly given by (116-117), where now we have set $B_1 = (\alpha+1)B$ and $B_2 = (\alpha - 1)B$ while $E_1,\ E_2$ are still given by $E_1 = -E_2 = -E$. The second set of equations is similar and we have

$$\begin{aligned} i\psi_{3,t} \quad &+ \quad \frac{1}{2M}\left[\psi_{3,xx} + \psi_{3,yy} - ieB(x\ \psi_{3,y} - y\ \psi_{3,x}) - \frac{e^2B^2}{4}(x^2+y^2)\psi_3 + eB_3\psi_3\right] \\ &+ \quad \frac{eE_3}{4M^2}\left(\psi_{4,x} + i\frac{eB}{2}y\ \psi_4\right) + \frac{e\tilde{E}_3}{4M^2}\left(i\psi_{4,y} + \frac{eB}{2}x\ \psi_4\right) = 0, \qquad (146) \\ i\psi_{4,t} \quad &+ \quad \frac{1}{2M}\left[\psi_{4,xx} + \psi_{4,yy} - ieB(x\ \psi_{4,y} - y\ \psi_{4,x}) - \frac{e^2B^2}{4}(x^2+y^2)\psi_4 + eB_4\psi_4\right] \\ &+ \quad \frac{eE_4}{4M^2}\left(\psi_{3,x} + i\frac{eB}{2}y\ \psi_3\right) + \frac{e\tilde{E}_4}{4M^2}\left(i\psi_{3,y} + \frac{eB}{2}x\ \psi_3\right) = 0, \qquad (147) \end{aligned}$$

where we have set $B_3 = (\beta+1)B$, $B_4 = (\beta-1)B$, $E_3 = -E_4 = -E$ and $\tilde{E}_3 = \tilde{E}_4 = E$. The prolongation method applied to this system and the associated complex conjugated equations leads to a vector field of the form

$$v = \xi_1 \partial_t + \xi_2 \partial_x + \xi_3 \partial_y + \sum_{\rho=1}^{4} \Phi_\rho \partial_{\psi_\rho} + \sum_{\rho=1}^{4} \bar{\Phi}_\rho \partial_{\bar{\psi}_\rho}, \tag{148}$$

where ξ_j $(j = 1,2,3)$ and Φ_ρ $(\rho = 1,\ldots,4)$ are functions dependent on t, x, y, ψ_ρ and $\bar{\psi}_\rho$. We get the solutions

$$\xi_1 = \delta_1, \qquad \xi_2(t,y) = -\delta_2 y + \delta_3, \qquad \xi_3(t,x) = \delta_2 x + \delta_4, \tag{149}$$

$$\begin{aligned}
\Phi_1(t,x,y) &= A_0(t,x,y) + A_1(x,y)\psi_1 + f(t)\psi_3, &(150)\\
\Phi_2(t,x,y) &= C_0(t,x,y) + C_2(x,y)\psi_2 + f(t)\psi_4, &(151)\\
\Phi_3(t,x,y) &= D_0(t,x,y) + g(t)\psi_1 + D_3(x,y)\psi_3, &(152)\\
\Phi_4(t,x,y) &= F_0(t,x,y) + g(t)\psi_2 + F_4(x,y)\psi_4. &(153)
\end{aligned}$$

The functions A_0, C_0, D_0 and F_0 are again arbitrary and such that they satisfy the equation (145) with $\Psi = (A_0, C_0, D_0, F_0)^t$. The other functions in (150) are given by

$$\begin{aligned}
A_1(x,y) &= -i\frac{eB}{2}(\delta_4 x - \delta_3 y) - i\frac{\delta_2}{2} + (\delta_7 + i\delta_5), &(154)\\
C_2(x,y) &= -i\frac{eB}{2}(\delta_4 x - \delta_3 y) + i\frac{\delta_2}{2} + (\delta_7 + i\delta_5), &(155)\\
D_3(x,y) &= -i\frac{eB}{2}(\delta_4 x - \delta_3 y) - i\frac{\delta_2}{2} + (\delta_8 + i\delta_6), &(156)\\
F_4(x,y) &= -i\frac{eB}{2}(\delta_4 x - \delta_3 y) + i\frac{\delta_2}{2} + (\delta_8 + i\delta_6) &(157)
\end{aligned}$$

and the functions $f(t)$ and $g(t)$ are obtained as

$$f(t) = (\delta_9 - i\delta_{11})e^{i\omega_{\alpha\beta}t}, \qquad g(t) = (\delta_{10} - i\delta_{12})e^{-i\omega_{\alpha\beta}t}. \tag{158}$$

with $\omega_{\alpha\beta} = \omega(\alpha-\beta) = \frac{eB}{2M}(\alpha-\beta)$.

Let us here comment on this last solution. The prolongation method has been applied to the set of equations (116-117) and (146-147) for arbitrary values of B_1, B_2, B_3 and B_4, and lead to the following sets of equations for f and g:

$$i\frac{df}{dt} = \frac{e}{2M}(B_3 - B_1)f = \frac{e}{2M}(B_4 - B_2)f \tag{159}$$

and

$$i\frac{dg}{dt} = \frac{e}{2M}(B_1 - B_3)g = \frac{e}{2M}(B_2 - B_4)g. \tag{160}$$

They are always compatible in the case under consideration, i.e. the one associated to the Hamiltonian (144), and we get the explicit solution (158). In this context, a particular constant solution is obtained when $\omega_{\alpha\beta} = 0$ or equivalently when $\alpha = \beta$. But if $(B_3 - B_1) \neq (B_4 - B_2)$, the equations (159) and (160) admit the trivial solution $f(t) = g(t) = 0$ which is a case that will be considered later.

Let us insist on the fact that since we are here in the case where f and g are given by (158), we will get symmetries expressed by odd generators that will satisfy structure relations corresponding to a superalgebra.

The infinitesimal generators of the finite dimensional Lie algebra of symmetries may be directly obtained from (148) with the preceding values (149-153) but once again to be able to get the finite transformations of symmetries, we have to express the vector fields in terms of the real variables u_ρ, ν_ρ, such that $\psi_\rho = u_\rho e^{i\nu_\rho}$ $(\rho = 1, \ldots, 4)$. We thus get the following basis of generators:

$$X_1 = \partial_t - \frac{\omega_{\alpha\beta}}{2}[(\partial_{\nu_3} - \partial_{\nu_1}) + (\partial_{\nu_4} - \partial_{\nu_2})], \tag{161}$$

$$X_2 = (x\partial_y - y\partial_x) - \frac{1}{2}[(\partial_{\nu_1} + \partial_{\nu_3}) - (\partial_{\nu_2} + \partial_{\nu_4})], \tag{162}$$

$$X_3 = \partial_x + \frac{eB}{2}\, y \sum_{\rho=1}^{4} \partial_{\nu_\rho}, \tag{163}$$

$$X_4 = \partial_y - \frac{eB}{2}\, x \sum_{\rho=1}^{4} \partial_{\nu_\rho}, \tag{164}$$

$$X_5 = \sum_{\rho=1}^{4} \partial_{\nu_\rho}, \tag{165}$$

$$X_6 = \sum_{\rho=1}^{4} u_\rho \partial_{u_\rho}, \tag{166}$$

$$\begin{aligned} X_7 &= \cos(\omega_{\alpha\beta}t - \nu_3 - \nu_1)u_3\partial_{u_1} + \frac{u_3}{u_1}\sin(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_1} \\ &+ \cos(\omega_{\alpha\beta}t - \nu_4 - \nu_2)u_4\partial_{u_2} + \frac{u_4}{u_2}\sin(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_2}, \end{aligned} \tag{167}$$

$$\begin{aligned} X_8 &= \cos(\omega_{\alpha\beta}t - \nu_3 - \nu_1)u_1\partial_{u_3} - \frac{u_1}{u_3}\sin(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_3} \\ &+ \cos(\omega_{\alpha\beta}t - \nu_4 - \nu_2)u_2\partial_{u_4} - \frac{u_2}{u_4}\sin(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_4}, \end{aligned} \tag{168}$$

$$\begin{aligned} X_9 &= (u_3\partial_{u_3} - u_1\partial_{u_1}) + (u_4\partial_{u_4} - u_2\partial_{u_2}), && (169)\\ X_{10} &= \sin(\omega_{\alpha\beta}t - \nu_3 - \nu_1)u_3\partial_{u_1} - \frac{u_3}{u_1}\cos(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_1} \\ &+ \sin(\omega_{\alpha\beta}t - \nu_4 - \nu_2)u_4\partial_{u_2} - \frac{u_4}{u_2}\cos(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_2}, && (170)\\ X_{11} &= -\sin(\omega_{\alpha\beta}t - \nu_3 - \nu_1)u_1\partial_{u_3} - \frac{u_1}{u_3}\sin(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_3} \\ &- \sin(\omega_{\alpha\beta}t - \nu_4 - \nu_2)u_2\partial_{u_4} - \frac{u_2}{u_4}\cos(\omega_{\alpha\beta}t - \nu_3 - \nu_1)\partial_{\nu_4}, && (171)\\ X_{12} &= (\partial_{\nu_3} - \partial_{\nu_1}) + (\partial_{\nu_4} - \partial_{\nu_2}). && (172) \end{aligned}$$

The generators X_j $(j = 1, \ldots, 6)$ satisfy the same commutation relations than the ones satisfied by the generators (122-127). The other generators X_j $(j = 7, \ldots, 12)$ form an algebra isomorphic to $su(2)^{\mathbb{C}}$ as in the Pauli case. The corresponding Lie algebra of symmetries of (145) is thus isomorphic to $(so(2) \dot{+} h(2)) \oplus su(2)^{\mathbb{C}} \oplus \{X_1, X_6\}$.

5.1 A Lie superalgebra of symmetries

The integration of the preceding vector fields leads to the corresponding finite transformations on the space-time coordinates and wave functions. We get, for X_1, the invariance under time translation such that

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = \begin{pmatrix} e^{\frac{i}{2}\omega_{\alpha\beta}\lambda_1}\sigma_0 & 0 \\ 0 & e^{-\frac{i}{2}\omega_{\alpha\beta}\lambda_1}\sigma_0 \end{pmatrix} \Psi(\tilde{t} - \lambda_1, \tilde{x}, \tilde{y}). \tag{173}$$

The integration of the vector field X_2 implies

$$\tilde{t} = t, \qquad \tilde{x} = x\cos\lambda_2 - y\sin\lambda_2, \quad \tilde{y} = x\sin\lambda_2 + y\cos\lambda_2 \tag{174}$$

and

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = \begin{pmatrix} e^{-\frac{i}{2}\lambda_2} & 0 & 0 & 0 \\ 0 & e^{\frac{i}{2}\lambda_2} & 0 & 0 \\ 0 & 0 & e^{-\frac{i}{2}\lambda_2} & 0 \\ 0 & 0 & 0 & e^{\frac{i}{2}\lambda_2} \end{pmatrix} \Psi(\tilde{t}, \tilde{x}\cos\lambda_2 + \tilde{y}\sin\lambda_2, -\tilde{x}\sin\lambda_2 + \tilde{y}\cos\lambda_2). \tag{175}$$

The integration of the vector fields X_3 and X_4 implies the invariance under space translations such that

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = e^{\frac{i}{2}eB\tilde{y}\lambda_3}\Psi(\tilde{t}, \tilde{x} - \lambda_3, \tilde{y}) \tag{176}$$

and

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = e^{-\frac{i}{2}eB\tilde{x}\lambda_4}\Psi(\tilde{t}, \tilde{x}, \tilde{y} - \lambda_4). \tag{177}$$

The integration of the vector fields X_5 and X_6 lead to phase and scale transformations of the wave functions

$$\tilde{\Psi}(t,x,y) = e^{i\lambda_5}\Psi(t,x,y) \tag{178}$$

and

$$\tilde{\Psi}(t,x,y) = e^{\lambda_6}\Psi(t,x,y). \tag{179}$$

The remaining vector fields X_j $(j = 7, \dots 12)$ lead to the transformations

$$\begin{aligned}
\tilde{\Psi}(t,x,y) &= \begin{pmatrix} \sigma_0 & \lambda_7 e^{i\omega_{\alpha\beta}t}\sigma_0 \\ 0 & \sigma_0 \end{pmatrix} \Psi(t,x,y), & (180)\\
\tilde{\Psi}(t,x,y) &= \begin{pmatrix} \sigma_0 & 0 \\ \lambda_8 e^{-i\omega_{\alpha\beta}t}\sigma_0 & \sigma_0 \end{pmatrix} \Psi(t,x,y), & (181)\\
\tilde{\Psi}(t,x,y) &= \begin{pmatrix} e^{-\lambda_9}\sigma_0 & 0 \\ 0 & e^{\lambda_9}\sigma_0 \end{pmatrix} \Psi(t,x,y), & (182)\\
\tilde{\Psi}(t,x,y) &= \begin{pmatrix} \sigma_0 & -i\lambda_{10} e^{i\omega_{\alpha\beta}t}\sigma_0 \\ 0 & \sigma_0 \end{pmatrix} \Psi(t,x,y), & (183)\\
\tilde{\Psi}(t,x,y) &= \begin{pmatrix} \sigma_0 & 0 \\ -i\lambda_{11} e^{-i\omega_{\alpha\beta}t}\sigma_0 & \sigma_0 \end{pmatrix} \Psi(t,x,y), & (184)\\
\tilde{\Psi}(t,x,y) &= \begin{pmatrix} e^{-i\lambda_{12}}\sigma_0 & 0 \\ 0 & e^{i\lambda_{12}}\sigma_0 \end{pmatrix} \Psi(t,x,y). & (185)
\end{aligned}$$

A specific matrix realization of the symmetry generators is obtained from these transformations, when they are developped at first order in the parameter λ_i. After some linear combinations and redefinitions, we get the following generators

$$\begin{aligned}
X_1 &= \partial_t\,\mathbb{I} - i\frac{\omega_{\alpha\beta}}{2}\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \;\; X_2 = (x\partial_y - y\partial_x)\,\mathbb{I} + \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, & (186)\\
X_3 &= \left(\partial_x - i\frac{eB}{2}\,y\right)\mathbb{I}, \quad X_4 = \left(\partial_y + i\frac{eB}{2}\,x\right)\mathbb{I}, \quad X_5 = -iX_6 = -i\mathbb{I}, & (187)\\
\mathbb{T}_+(t) &= X_7 = -iX_{10} = e^{i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & \sigma_0 \\ 0 & 0 \end{pmatrix}, & (188)\\
\mathbb{T}_-(t) &= X_8 = -iX_{11} = e^{-i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & 0 \\ \sigma_0 & 0 \end{pmatrix}, & (189)\\
\mathbb{Y} &= X_9 = -iX_{12} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, & (190)
\end{aligned}$$

where $\mathbb{I}$ is the 4×4 identity matrix. We see that to X_1, we can associate the generator

$$\mathbb{H} = i\partial_t\,\mathbb{I} + \frac{\omega_{\alpha\beta}}{2}\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \tag{191}$$

which will be related later to a new Hamiltonian refering to a symmetrized version of the JC Hamiltonian. The generator X_2 corresponds to the total angular momentum

$$\mathbb{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \tag{192}$$

and X_3 and X_4 may be combined to give

$$\mathbb{A}_- = \begin{pmatrix} A_- & 0 \\ 0 & A_- \end{pmatrix}, \quad \mathbb{A}_+ = \begin{pmatrix} A_+ & 0 \\ 0 & A_+ \end{pmatrix}, \tag{193}$$

where A_- and A_+ are given in (140-141).

The generators $\mathbb{T}_+(t), \mathbb{T}_-(t)$ and $\mathbb{Y}$ may now be associated to odd generators and, together with $\mathbb{A}_-,\ \mathbb{A}_+$ and $\mathbb{I}$, form a $sh(2/2)$ superalgebra. As in the cases of the SUSY harmonic oscillator and the Pauli systems, odd products may be formed between $\{\mathbb{A}_-,\ \mathbb{A}_+\}$ and $\{\mathbb{T}_+(t),\ \mathbb{T}_-(t)\}$ and among the possible ones we get the following generators:

$$\begin{aligned} \mathbb{S}_-(t) &= \sqrt{\tilde{\omega}}\mathbb{A}_+\mathbb{T}_-(t) = \sqrt{\omega}e^{-i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & 0 \\ A_+ & 0 \end{pmatrix}, & (194) \\ \mathbb{S}_+(t) &= \sqrt{\tilde{\omega}}\mathbb{A}_-\mathbb{T}_+(t) = \sqrt{\tilde{\omega}}e^{i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & A_- \\ 0 & 0 \end{pmatrix}, & (195) \end{aligned}$$

and

$$\begin{aligned} \mathbb{U}_-(t) &= \sqrt{\tilde{\omega}}\mathbb{A}_-\mathbb{T}_-(t) = \sqrt{\tilde{\omega}}e^{-i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & 0 \\ A_- & 0 \end{pmatrix}, & (196) \\ \mathbb{U}_+(t) &= \sqrt{\tilde{\omega}}\mathbb{A}_+\mathbb{T}_+(t) = \sqrt{\tilde{\omega}}e^{i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & A_+ \\ 0 & 0 \end{pmatrix}, & (197) \end{aligned}$$

which satisfy the anticommutation relations

$$\{\mathbb{S}_-(t), \mathbb{S}_+(t)\} = \mathbb{H}_0 + \frac{\tilde{\omega}}{2}\mathbb{Y} = \tilde{\omega}\begin{pmatrix} A_-A_+ & 0 \\ 0 & A_+A_- \end{pmatrix} \tag{198}$$

and

$$\{\mathbb{U}_-(t), \mathbb{U}_+(t)\} = \mathbb{H}_0 - \frac{\tilde{\omega}}{2}\mathbb{Y} = \tilde{\omega}\begin{pmatrix} A_+A_- & 0 \\ 0 & A_-A_+ \end{pmatrix}, \tag{199}$$

where we have defined

$$\mathbb{H}_0 = \tilde{\omega}\begin{pmatrix} \left(A_+A_- + \frac{1}{2}\right) & 0 \\ 0 & \left(A_+A_- + \frac{1}{2}\right) \end{pmatrix}. \tag{200}$$

The two components of $\mathbb{H}_0$ are not related to $\mathbb{H}_{JC}$ since they have only diagonal terms. The non-zero commutation relations between the generators $\mathbb{S}_\pm(t)$ and $\mathbb{U}_\pm(t)$ are given by

$$\{\mathbb{S}_-, \mathbb{U}_+\} = -2i\mathbb{C}_+ = \tilde{\omega}(\mathbb{A}_+)^2, \qquad \{\mathbb{S}_+, \mathbb{U}_-\} = -2i\mathbb{C}_- = \tilde{\omega}(\mathbb{A}_-)^2. \tag{201}$$

	$\mathbb{H}_0$	$\mathbb{C}_-$	$\mathbb{C}_+$	$\mathbb{Y}$	$\mathbb{S}_-$	$\mathbb{S}_+$	$\mathbb{U}_-$	$\mathbb{U}_+$
$\mathbb{H}_0$	0	$-2\tilde{\omega}\mathbb{C}_-$	$2\tilde{\omega}\mathbb{C}_+$	0	$\tilde{\omega}\mathbb{S}_-$	$-\tilde{\omega}\mathbb{S}_+$	$-\tilde{\omega}\mathbb{U}_-$	$\tilde{\omega}\mathbb{U}_+$
$\mathbb{C}_-$	$2\tilde{\omega}\mathbb{C}_-$	0	$-\tilde{\omega}\mathbb{H}_0$	0	$i\tilde{\omega}\mathbb{U}_-$	0	0	$i\tilde{\omega}\mathbb{S}_+$
$\mathbb{C}_+$	$-2\tilde{\omega}\mathbb{C}_+$	$\tilde{\omega}\mathbb{H}_0$	0	0	0	$-i\tilde{\omega}\mathbb{U}_+$	$-i\tilde{\omega}\mathbb{S}_-$	0
$\mathbb{Y}$	0	0	0	0	$-2\mathbb{S}_-$	$2\mathbb{S}_+$	$-2\mathbb{U}_-$	$2\mathbb{U}_+$
$\mathbb{S}_-$	$-\tilde{\omega}\mathbb{S}_-$	$-i\tilde{\omega}\mathbb{U}_-$	0	$2\mathbb{S}_-$	0	$\mathbb{H}_0 + \tilde{\omega}\mathbb{Y}/2$	0	$-2i\mathbb{C}_+$
$\mathbb{S}_+$	$\tilde{\omega}\mathbb{S}_+$	0	$i\tilde{\omega}\mathbb{U}_+$	$-2\mathbb{S}_+$	$\mathbb{H}_0 + \tilde{\omega}\mathbb{Y}/2$	0	$-2i\mathbb{C}_-$	0
$\mathbb{U}_-$	$\tilde{\omega}\mathbb{U}_-$	0	$i\tilde{\omega}\mathbb{S}_-$	$2\mathbb{U}_-$	0	$-2i\mathbb{C}_-$	0	$\mathbb{H}_0 - \tilde{\omega}\mathbb{Y}/2$
$\mathbb{U}_+$	$-\tilde{\omega}\mathbb{U}_+$	$-i\tilde{\omega}\mathbb{S}_+$	0	$-2\mathbb{U}_+$	$-2i\mathbb{C}_+$	0	$\mathbb{H}_0 - \tilde{\omega}\mathbb{Y}/2$	0

Table 6: Super-commutation relations of an $osp(2/2)$ superalgebra.

Now the set $\{\mathbb{H}, \mathbb{H}_0, \mathbb{C}_\pm, \mathbb{Y}, \mathbb{S}_\pm(t), \mathbb{U}_\pm(t), \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(t)\}$ closes a superalgebra. We see that $\mathbb{H}$ commutes with all the generators. The commutation relations between the generators $\mathbb{H}_0, \mathbb{C}_\pm, \mathbb{Y}, \mathbb{S}_\pm(t)$ and $\mathbb{U}_\pm(t)$ are given in table 6. These last generators form a superalgebra isomorphic to $osp(2/2)$. The non-zero super-commutation relations between the generators $\mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(t)$, are now given by

$$[\mathbb{J}, \mathbb{A}_+] = \mathbb{A}_+, \qquad [\mathbb{J}, \mathbb{A}_-] = -\mathbb{A}_- \tag{202}$$

and

$$[\mathbb{A}_-, \mathbb{A}_+] = \mathbb{I} = \{\mathbb{T}_-(t), \mathbb{T}_+(t)\}, \tag{203}$$

leading to the superalgebra $so(2) \dot{+} sh(2/2)$. Finally the super-commutation relations between the two sets are presented in table 7. So we find a structure isomorphic to the superalgebra $(so(2) \dot{+} osp(2/2)) \dot{+} sh(2/2)$. Let us insist on the fact that the existence of such a superalgebra does not implies the presence of supersymmetries for the original Hamiltonian (144). Indeed, no Q-type supercharges may be constructed from the preceding symmetries. In the last subsections, SUSY JC models will be constructed and the symmetries and supersymmetries will be given.

5.2 A supersymmetric JC model

The generator (191) when acting on the space of solutions of (145), corresponds to a symmetrized version of the Hamiltonian (144) given by

$$\mathbb{H} = H_T + \frac{\omega_{\alpha\beta}}{2}\mathbb{Y} = \begin{pmatrix} H_{JC} - \frac{eB}{4M}(\alpha+\beta)\sigma_0 & 0 \\ 0 & H_{JC} - \frac{eB}{4M}(\alpha+\beta)\sigma_0 \end{pmatrix}. \tag{204}$$

	$\mathbb{H}_0$	$\mathbb{C}_-$	$\mathbb{C}_+$	$\mathbb{Y}$	$\mathbb{S}_-$	$\mathbb{S}_+$	$\mathbb{U}_-$	$\mathbb{U}_+$
$\mathbb{J}$	0	$-2\mathbb{C}_-$	$2\mathbb{C}_+$	0	$\mathbb{S}_-$	$-\mathbb{S}_+$	$-\mathbb{U}_-$	$\mathbb{U}_+$
$\mathbb{A}_-$	$\tilde{\omega}\mathbb{A}_-$	0	$i\tilde{\omega}\mathbb{A}_+$	0	$\sqrt{\tilde{\omega}}\mathbb{T}_-$	0	0	$\sqrt{\tilde{\omega}}\mathbb{T}_+$
$\mathbb{A}_+$	$-\tilde{\omega}\mathbb{A}_+$	$-i\tilde{\omega}\mathbb{A}_-$	0	0	0	$-\sqrt{\tilde{\omega}}\mathbb{T}_+$	$-\sqrt{\tilde{\omega}}\mathbb{T}_-$	0
$\mathbb{I}$	0	0	0	0	0	0	0	0
$\mathbb{T}_-$	0	0	0	$2\mathbb{T}_-$	0	$\sqrt{\tilde{\omega}}\mathbb{A}_-$	0	$\sqrt{\tilde{\omega}}\mathbb{A}_+$
$\mathbb{T}_+$	0	0	0	$-2\mathbb{T}_+$	$\sqrt{\tilde{\omega}}\mathbb{A}_+$	0	$\sqrt{\tilde{\omega}}\mathbb{A}_-$	0

Table 7: Commutation relations between $(so(2)\dot{+}osp(2/2))$ and $sh(2/2)$.

From the preceding results, it is easy to show that the symmetries of this new Hamiltonian are given by the set $\{\mathbb{H}, \mathbb{H}_0, \mathbb{C}_\pm, \mathbb{Y}, \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(0)\}$, all of these generators being time independent. Indeed, $\mathbb{H}$ can be seen as a particular H_T as given in (144) where $\alpha = \beta$ and thus $\omega_{\alpha\beta} = 0$.

It admits also the supersymmetries $\mathbb{S}_\pm(0)$ and $\mathbb{U}_\pm(0)$ which are now time independent and satisfy again (198) and (199). None of them are the supercharges of $\mathbb{H}$.

Let us now show that, for a specific value of $\alpha + \beta$, the symmetrized Hamiltonian $\mathbb{H}$ can be made supersymmetric. Indeed, if we take $(\alpha + \beta) = -\frac{eE}{8M^2B}$, we can define [9]

$$\mathbb{Q}_+ = \frac{\bar{\kappa}}{2\sqrt{\tilde{\omega}}}\mathbb{T}_+(0) + \sqrt{\tilde{\omega}} \begin{pmatrix} 0 & (Q_+ - Q_-) \\ 0 & 0 \end{pmatrix} \tag{205}$$

and

$$\mathbb{Q}_- = \frac{\kappa}{2\sqrt{\tilde{\omega}}}\mathbb{T}_-(0) + \sqrt{\tilde{\omega}} \begin{pmatrix} 0 & 0 \\ (Q_- - Q_+) & 0 \end{pmatrix}. \tag{206}$$

We thus have

$$\{\mathbb{Q}_+, \mathbb{Q}_-\} = \mathbb{H}, \qquad [\mathbb{H}, \mathbb{Q}_\pm] = 0. \tag{207}$$

The time independent generators $\mathbb{H}, \mathbb{Y}, \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}$ and $\mathbb{Q}_\pm$ form a superalgebra of supersymmetries of $\mathbb{H}$. The additionnal super-commutation relations are

$$[\mathbb{Y}, \mathbb{Q}_\pm] = \pm 2\mathbb{Q}_\pm. \tag{208}$$

If we include the generators $\mathbb{T}_\pm(0)$ as symmetries of $\mathbb{H}$, we get the following superalgebra $\{\mathbb{H}, \mathbb{Y}, \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{Q}_\pm, \mathbb{Q}_0, \mathbb{T}_\pm(0)\}$, where

$$\mathbb{Q}_0 = \begin{pmatrix} (Q_+ - Q_-) & 0 \\ 0 & (Q_+ - Q_-) \end{pmatrix}. \tag{209}$$

Indeed we have

$$\{\mathbb{T}_+(0), \mathbb{Q}_-\} = \frac{\kappa}{2\sqrt{\tilde{\omega}}}\mathbb{I} - \sqrt{\tilde{\omega}}\mathbb{Q}_0, \qquad \{\mathbb{T}_-(0), \mathbb{Q}_+\} = \frac{\bar{\kappa}}{2\sqrt{\tilde{\omega}}}\mathbb{I} + \sqrt{\tilde{\omega}}\mathbb{Q}_0. \tag{210}$$

This superalgebra may be written as $(\{\mathbb{H}, \mathbb{Y}, \mathbb{Q}_\pm\} \oplus \{\mathbb{J}\}) \dot{+} \{\mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(0), \mathbb{Q}_0\}$.

In the approach of Andreev and Lerner [9], the preceding Hamiltonian $\mathbb{H}$ has been generalized to $\mathbb{H}(\varphi)$ where φ is an arbitrary phase. Indeed, $\mathbb{H}(\varphi)$ is block diagonal where, up to the addition of a multiple of the identity, the first block is the JC Hamiltonian (114) and the second one is obtained from it by changing $\mathcal{A} \mapsto e^{-i\varphi}\mathcal{A}$ and $\mathcal{A}^\dagger \mapsto e^{i\varphi}\mathcal{A}^\dagger$. With respect to our approach, it is associated to the original set of equations (116-117) and the new set (146-147) where $E_3 = -\tilde{E}_3 = Ee^{i\varphi}$ and $E_4 = \tilde{E}_4 = Ee^{-i\varphi}$. The algebra of symmetries is the same as for the case $\varphi = 0$ studied before, so all the results about the existence of supersymmetry transformations remain valid. The only changes are in the following generators

$$\mathbb{T}_+(\varphi, t) = e^{i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \\ 0 & 0 \end{pmatrix}, \mathbb{T}_-(\varphi, t) = e^{-i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} & 0 \end{pmatrix}. \tag{211}$$

It follows that the generators $\mathbb{S}_\mp(t)$ and $\mathbb{U}_\mp(t)$ given (194-195) and (196-197) in are now written as $\mathbb{S}_\mp(\varphi, t) = \sqrt{\tilde{\omega}}\mathbb{A}_\pm\mathbb{T}_\mp(\varphi, t)$ and $\mathbb{U}_\mp(\varphi, t) = \sqrt{\tilde{\omega}}\mathbb{A}_\mp\mathbb{T}_\mp(\varphi, t)$.

The supercharges are found to be

$$\mathbb{Q}_+(\varphi) = \frac{\bar{\kappa}}{2\sqrt{\tilde{\omega}}}\mathbb{T}_+(\varphi, 0) + \sqrt{\tilde{\omega}}\begin{pmatrix} 0 & (e^{i\varphi}Q_+ - Q_-) \\ 0 & 0 \end{pmatrix}, \tag{212}$$

$$\mathbb{Q}_-(\varphi) = \frac{\kappa}{2\sqrt{\tilde{\omega}}}\mathbb{T}_-(\varphi, 0) + \sqrt{\tilde{\omega}}\begin{pmatrix} 0 & 0 \\ (e^{-i\varphi}Q_- - Q_+) & 0 \end{pmatrix} \tag{213}$$

and satisfy

$$\{\mathbb{Q}_+(\varphi), \mathbb{Q}_-(\varphi)\} = \mathbb{H}(\varphi), \qquad [\mathbb{H}(\varphi), \mathbb{Q}_\pm(\varphi)] = 0. \tag{214}$$

The last generator that is modified is

$$\mathbb{Q}_0(\varphi) = \begin{pmatrix} (Q_+ - Q_-) & 0 \\ 0 & (e^{i\varphi}Q_+ - e^{-i\varphi}Q_-) \end{pmatrix}. \tag{215}$$

5.3 The usual supersymmetric structure

Another SUSY version of the JC model may be deduced from our preceding considerations. Let us refer it as the standard or strong coupling limit one in reference with the literature

[20, 21, 22]. If we start again with the system of equations (116-117) and (146-147) for which the symmetries have been determined for arbitrary values of the parameters B_1, B_2, B_3, B_4, we can in particular take $B_1 = -B_2 = B$, while $B_3 = -3B$ and $B_4 = -B$. This is the case where the equations (159) and (160) admit the trivial solution $f(t) = g(t) = 0$ and such that no symmetries associated with odd generators appear. This means that, by the prolongation method, it will be impossible to get a superalgebra of symmetries.

Meanwhile, if we choose $E_2 = -E_1 = E_4 = -E_3 = \tilde{E}_3 = \tilde{E}_4 = E = M/\sqrt{2eB}$, it is possible to write the evolution equations (116-117) and (146-147) as

$$(i\partial_t - \mathbb{H}_{JC})\Psi(t,x,y) = 0, \tag{216}$$

where $\mathbb{H}_{JC}$ has the standard SUSY form[23]

$$\mathbb{H}_{JC} = \tilde{\omega} \begin{pmatrix} \tilde{\mathcal{A}}^\dagger \tilde{\mathcal{A}} & 0 \\ 0 & \tilde{\mathcal{A}} \tilde{\mathcal{A}}^\dagger \end{pmatrix}, \tag{217}$$

with

$$\tilde{\mathcal{A}} = \mathcal{A} + i\sigma_+, \qquad \tilde{\mathcal{A}}^\dagger = \mathcal{A}^\dagger - i\sigma_-. \tag{218}$$

These last operators satisfy

$$[\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^\dagger] = \sigma_0 + \sigma_3. \tag{219}$$

Note that the two components of the Hamiltonian $\mathbb{H}_{JC}$ are closely related to the Hamiltonian (114). Indeed, we have

$$\mathbb{H}_{JC} = \begin{pmatrix} H_{JC} & 0 \\ 0 & H_{JC} + \tilde{\omega} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}, \tag{220}$$

when $\kappa = i\tilde{\omega}$. The standard supercharges are given by

$$\tilde{\mathcal{Q}}_+ = \tilde{\omega}^{1/2} \begin{pmatrix} 0 & \tilde{\mathcal{A}}^\dagger \\ 0 & 0 \end{pmatrix}, \qquad \tilde{\mathcal{Q}}_- = \tilde{\omega}^{1/2} \begin{pmatrix} 0 & 0 \\ \tilde{\mathcal{A}} & 0 \end{pmatrix}, \tag{221}$$

satisfying the following relations

$$\mathbb{H}_{JC} = \{\tilde{\mathcal{Q}}_+, \tilde{\mathcal{Q}}_-\}, \qquad (\tilde{\mathcal{Q}}_\pm)^2 = 0, \qquad [\mathbb{H}_{JC}, \tilde{\mathcal{Q}}_\pm] = 0. \tag{222}$$

Again these supercharges can not be obtained from the product of symmetries determined by the prolongation method.

Let us finally mention that such a Hamiltonian is a particular case of a matrix SUSY one where the quantities $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^\dagger$ would correspond to elements of the algebra $h(2) \oplus su(2)$, that is linear combinations of the generators of this algebra. Different assumptions may thus be imposed on the commutator $[\mathcal{A}, \mathcal{A}^\dagger]$ [22]. In the canonical case, that is the case where the commutator is a multiple of the identity, the prolongation method reproduce all the dynamical supersymmetries. This was the case for the SUSY harmonic oscillator and the Pauli Hamiltonians. In the non-canonical case, like for example the JC model, the supercharges are not obtained from the prolongation method but may be constructed by the standard structure of the Hamiltonian like in (217) for the JC model.

6 Conclusion

We have shown that the prolongation method used for finding symmetries of classical as well as quantum mechanical systems may be useful to determine the supersymmetries of SUSY quantum mechanical systems. We took simple examples, like the SUSY harmonic oscillator and the Pauli equations to improve the method. Indeed, we already knew the kind of kinematical and dynamical superalgebras we were searching for. This was very helpful to be able to get new results on the JC model. First we determine the Lie algebra of symmetries for the usual 2 by 2 matrix model. Second, we gave the symmetry superalgebra for a generalized version which is a amplification of the usual JC model to a 4 by 4 matrix representation. Finally, two ways of getting SUSY versions where given. In the first case the supersymmetry was present only if we admit a specific shifting in the JC Hamiltonian. In the second case, the supersymmetry appeared due to the fact that the amplification of the JC model is similar to the one of the SUSY harmonic oscillator but to do it was necessary to take the coupling constant between the electromagnetic field and the atom as a linear function of the frequency of these fields. In all these cases, the detuning between the electromagnetic field and atom frequencies has been assumed to be equal to zero.

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